Some applications of light-cone superspace

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$\mathcal{N}=4$ supersymmetric Yang–Mills theory

Field content

- 6 real scalars, φ^{AB} $\quad [\bar{\varphi}_{AB} = (\varphi^{AB})^* = \frac{1}{2}\varepsilon_{ABCD}\varphi^{CD}]$
- 4 Weyl fermions, λ^A_α, $\bar{\lambda}^{\dot{\alpha}}_A$ $\quad A, B = 1, \dots, 4$
- 1 vector, A_μ

All the fields are in the adjoint of the gauge group G.

$\mathcal{N}=4$ $\rightarrow$ Maximal supersymmetry in four dimensions. SU(4)$\sim$SO(6) R-symmetry group.

Action

$$\mathcal{S} = \int \mathrm{d}^4x \, \mathrm{Tr}\left[\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + D^\mu\bar{\varphi}_{AB}D_\mu\varphi^{AB} - 2i\,\lambda^A \not{D}\bar{\lambda}_A \right.$$
$$\left. -2\sqrt{2}\,g\Big([\lambda^A,\lambda^B]\bar{\varphi}_{AB} + [\bar{\lambda}_A,\bar{\lambda}_B]\varphi^{AB}\Big) - \frac{1}{2}g^2[\varphi^{AB},\varphi^{CD}][\bar{\varphi}_{AB},\bar{\varphi}_{CD}]\right]$$

The theory has a number of remarkable properties:

- UV finite and quantum conformally invariant
  $\rightarrow$ PSU(2,2|4) group of global symmetries
- Invariant under SL(2,$\mathbb{Z}$) S-duality
- Related to type IIB superstring theory by the AdS/CFT correspondence [G=SU(N)]
- Integrability (of the spectrum)

$\mathcal{N}=4$ SYM in the light-cone gauge

Light-cone coordinates:

$$x^\mu \to \begin{cases} x^\pm = \frac{1}{\sqrt{2}}\left(x^0 \pm x^3\right) \\ x = \frac{1}{\sqrt{2}}\left(x^1 + ix^2\right) \ , \ \bar{x} = \frac{1}{\sqrt{2}}\left(x^1 - ix^2\right) \end{cases}$$

Similarly for the gauge field: $A_\mu \ \to \ A_\pm \, , \ A \, , \ \bar{A}$

Light-cone gauge:

$A_- = 0$ [$\to$ no ghosts]

A_+ is eliminated using the equations of motion.

Fermions:

$\lambda^A_\alpha \to (\chi^A_{(+)}, \chi^A_{(-)})$ [1-component Grassmann variables]

$\chi^A_{(\pm)} = \mathcal{P}_\pm \lambda^A_\alpha \ , \quad \mathcal{P}_\pm = \frac{1}{2}\gamma^\pm\gamma^\mp$

$\chi^A_{(+)}$ is eliminated using the e.o.m. [$\chi_{(-)} \equiv \chi$]

Physical components:

$A \, , \ \bar{A} \, , \ \varphi^{AB} \, , \ \chi^A \, , \ \bar{\chi}_A$ [8 bosons + 8 fermions]

Supersymmetry transformations

$$\delta A = i\epsilon^A \bar{\chi}_A$$

$$\delta\varphi^{AB} = -i(\epsilon^A\chi^B - \epsilon^B\chi^A + \varepsilon^{ABCD}\bar{\epsilon}_C\bar{\chi}_D)$$

$$\delta\chi^A = \sqrt{2}\epsilon^A\partial_-\bar{A} - \sqrt{2}\bar{\epsilon}_B\partial_-\varphi^{AB}$$

Light-cone component action

$$
\begin{aligned}
\mathcal{S} = \int \mathrm{d}^4x\,\mathrm{Tr}\Big\{ & 2\bar{A}\Box A + \frac{1}{2}\,\bar{\varphi}_{AB}\Box\varphi^{AB} - \frac{2i}{\sqrt{2}}\,\bar{\chi}_A\frac{\Box}{\partial_-}\chi^A \\
& +g\Big[4i\,\frac{\bar{\partial}}{\partial_-}A[\partial_-\bar{A}, A] + i\,\frac{\bar{\partial}}{\partial_-}A[\partial_-\bar{\varphi}_{AB}, \varphi^{AB}] - i\,A[\bar{\partial}\bar{\varphi}_{AB}, \varphi^{AB}] \\
& -2\sqrt{2}\,\frac{\bar{\partial}}{\partial_-}A[\bar{\chi}_A, \chi^A] + 2\sqrt{2}\,A[\chi^A, \frac{\bar{\partial}}{\partial_-}\bar{\chi}_A] - 2\sqrt{2}\frac{\bar{\partial}}{\partial_-}\bar{\chi}_A[\bar{\chi}_B, \varphi^{AB}] \\
& +\mathrm{h.c.}\Big] \\
& +g^2\Big[4\frac{1}{\partial_-}[\partial_-A, \bar{A}]\frac{1}{\partial_-}[\partial_-\bar{A}, A] + [\varphi^{AB}, A][\bar{\varphi}_{AB}, \bar{A}] \\
& +\frac{1}{\partial_-}[\partial_-\bar{A}, A]\frac{1}{\partial_-}[\partial_-\bar{\varphi}_{AB}, \varphi^{AB}] + \frac{1}{\partial_-}[\partial_-A, \bar{A}]\frac{1}{\partial_-}[\partial_-\bar{\varphi}_{AB}, \varphi^{AB}] \\
& +\frac{1}{8}[\varphi^{AB}, \varphi^{CD}][\bar{\varphi}_{AB}, \bar{\varphi}_{CD}] + \frac{1}{4}\,\frac{1}{\partial_-}[\partial_-\bar{\varphi}_{AB}, \varphi^{AB}]\frac{1}{\partial_-}[\partial_-\bar{\varphi}_{CD}, \varphi^{CD}] \\
& -i2\sqrt{2}\,\frac{1}{\partial_-}[\bar{\chi}_A, \bar{A}][A, \chi^A] + i2\sqrt{2}\,\frac{1}{\partial_-}[\chi^A, A][\bar{\varphi}_{AB}, \chi^B] \\
& +i2\sqrt{2}\,\frac{1}{\partial_-}[\bar{\chi}_A, \bar{A}][\varphi^{AB}, \bar{\chi}_B] + i2\sqrt{2}\,\frac{1}{\partial_-}[\bar{\chi}_A, \varphi^{AB}][\bar{\varphi}_{BC}, \chi^C] \\
& +i2\sqrt{2}\,\frac{1}{\partial_-}[\partial_-A, \bar{A}]\frac{1}{\partial_-}[\bar{\chi}_A, \chi^A] + i2\sqrt{2}\,\frac{1}{\partial_-}[\partial_-\bar{A}, A]\frac{1}{\partial_-}[\bar{\chi}_A, \chi^A] \\
& +i\sqrt{2}\,\frac{1}{\partial_-}[\partial_-\bar{\varphi}_{AB}, \varphi^{AB}]\frac{1}{\partial_-}[\bar{\chi}_C, \chi^C] - 2\frac{1}{\partial_-}[\bar{\chi}_A, \chi^A]\frac{1}{\partial_-}[\bar{\chi}_B, \chi^B]\Big]\Big\}
\end{aligned}
$$

Light-cone superspace

Coordinates: $z = (x^+, x^-, x, \bar{x}; \theta^A, \bar{\theta}_A)$, $A = 1, \ldots .4$

$\longrightarrow$ Chiral coordinate: $y^- = x^- - \frac{1}{\sqrt{2}}\theta^A\bar{\theta}_A$

Supercharges $[q \equiv q_{(+)}]$

$$q^A = -\frac{\partial}{\partial\bar{\theta}_A} - \frac{i}{\sqrt{2}}\theta^A\partial_- \quad , \quad \bar{q}_A = \frac{\partial}{\partial\theta^A} + \frac{i}{\sqrt{2}}\bar{\theta}_A\partial_-$$

The other eight supercharges, $q^A_{(-)}$ and $\bar{q}_{A\,(-)}$, are non-linearly realised.

Super algebra

$$\{q^A_{(+)}, \bar{q}_{B\,(+)}\} = \sqrt{2}\delta^A{}_B\, p_-$$
$$\{q^A_{(-)}, \bar{q}_{B\,(-)}\} = \sqrt{2}\delta^A{}_B\, p_+ \equiv \sqrt{2}\delta^A{}_B H$$
$$\{q^A_{(-)}, \bar{q}_{B\,(+)}\} = \sqrt{2}\delta^A{}_B\, p$$

Chiral derivatives

$$d^A = -\frac{\partial}{\partial\bar{\theta}_A} + \frac{i}{\sqrt{2}}\theta^A\partial_- \quad , \quad \bar{d}_A = \frac{\partial}{\partial\theta^A} - \frac{i}{\sqrt{2}}\bar{\theta}_A\partial_-$$

$$\{d^A, \bar{d}_B\} = i\sqrt{2}\delta^A{}_B\,\partial_-$$

$\mathscr{N}=4$ light-cone superfield

$\mathscr{N}=4$ on-shell multiplet $\longrightarrow$ $(A\,,\ \bar{A}\,,\ \chi^A\,,\ \bar{\chi}_A\,,\ \varphi^{AB})$

All the physical component fields can be packaged into a single scalar superfield:

$\Phi = \Phi(x^+, x^-, x, \bar{x}, \theta^A, \bar{\theta}_A)$

The irreducible $\mathscr{N}=4$ superfield is defined by the two constraints

$d^A\Phi = 0\,,\ \bar{d}_A\bar{\Phi} = 0$ (chirality)

$\bar{d}_A\bar{d}_B\Phi = \frac{1}{2}\varepsilon_{ABCD}d^C d^D\bar{\Phi}$ (reality, a.k.a. "inside-out")

The superfield solving the constraints is

$$\Phi(x,\theta,\bar{\theta}) = -\frac{1}{\partial_-}A(y) - \frac{i}{\partial_-}\theta^A\bar{\chi}_A(y) + \frac{i}{\sqrt{2}}\theta^A\theta^B\bar{\varphi}_{AB}(y) + \frac{\sqrt{2}}{6}\theta^A\theta^B\theta^C\varepsilon_{ABCD}\chi^D(y) - \frac{1}{12}\theta^A\theta^B\theta^C\theta^D\varepsilon_{ABCD}\partial_-\bar{A}(y)$$

[where $y = (x, \bar{x}, x^+, y^- = x^- - \frac{i}{\sqrt{2}}\theta^A\bar{\theta}_A)$]

From the reality constraint

$$\bar{\Phi} = \frac{1}{48}\frac{\bar{d}^4}{\partial_-^2}\Phi$$

[where $\bar{d}^4 = \varepsilon^{ABCD}\bar{d}_A\bar{d}_B\bar{d}_C\bar{d}_D$]

$\mathcal{N}=4$ light-cone superspace action

$$
\begin{aligned}
\mathcal{S} \;=\; 72\int \mathrm{d}^4x \int \mathrm{d}^4\theta\, \mathrm{d}^4\bar{\theta}\ \mathrm{Tr}\Big\{ & -2\,\bar{\Phi}\,\frac{\Box}{\partial_-^2}\,\Phi \\
& + i\,\frac{8}{3}\,g\left(\frac{1}{\partial_-}\bar{\Phi}\,[\Phi,\bar{\partial}\Phi] \;+\; \frac{1}{\partial_-}\,\Phi\,[\bar{\Phi},\partial\bar{\Phi}]\right) \\
& + 2\,g^2\left(\frac{1}{\partial_-}[\Phi,\partial_-\Phi]\,\frac{1}{\partial_-}[\bar{\Phi},\partial_-\bar{\Phi}] \;+\; \frac{1}{2}[\Phi,\bar{\Phi}]\,[\Phi,\bar{\Phi}]\right)\Big\}
\end{aligned}
$$

It can be rewritten in terms of Φ only using the constraint

$$
\bar{\Phi} = \frac{1}{48}\,\frac{\bar{d}^4}{\partial_-^2}\,\Phi
$$

In this formulation both the $\mathcal{N}=4$ supersymmetry and the SU(4) R-symmetry are manifest, but the Lorentz invariance is not explicit.

The light-cone superspace formulation can be used to prove scale invariance of $\mathcal{N}=4$ SYM to all orders in perturbation theory: all the Green functions can be shown explicitly to be finite in the light-cone superspace formulation.

Deformations of $\mathcal{N}=4$ SYM

$\mathcal{N}=4$ SYM is maximally supersymmetric. It is interesting to look for theories with similar properties, but less supersymmetry.

Marginal deformations

$$\mathcal{S} = \int \mathrm{d}^4x \, \mathcal{L}_{\mathcal{N}=4} + \mathcal{O}(x)$$

where $\mathcal{O}(x)$ is an exactly marginal operator.

A special class of deformations are those characterised by the superpotential (in $\mathcal{N}=1$ superspace)

$$W = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, g \, \mathrm{Tr} \left\{ h \, \varepsilon_{IJK} \, \Phi^I [\Phi^J, \Phi^K]_* + k_I \left(\Phi^I\right)^3 \right\} + \mathrm{h.c.}$$

where the $*$-commutator is defined as

$$[\Phi^I, \Phi^J]_* = \Phi^I * \Phi^J - \Phi^J * \Phi^I$$

$$\Phi^I * \Phi^J = \mathrm{e}^{i\pi\beta_{IJ}} \Phi^I \Phi^J \quad \text{with} \quad \beta_{JI} = -\beta_{IJ}$$

In the special case $k_I = k$, $|\beta_{IJ}| = \beta$, $\forall I, J = 1,2,3$ it has been argued that the deformed theory can be rendered finite imposing a single relation among the parameters

$$\gamma(g, h, \beta, k, N) = 0$$

$\rightarrow$ 2-parameter family of exactly marginal deformations of $\mathcal{N}=4$ SYM

Deformations and AdS/CFT

The exactly marginal deformations of $\mathcal{N}=4$ SYM are interesting in the context of the AdS/CFT correspondence. They are expected to be dual to string theory on backgrounds of the form $\mathrm{AdS}_5 \times X_5$, with X_5 a five-dimensional Einstein manifold.

β-deformation

In the special case of the superpotential

$$W = \int \mathrm{d}^4x \, \mathrm{d}^2\theta \, g \, \mathrm{Tr} \left(\mathrm{e}^{i\pi\beta} \Phi^1 \Phi^2 \Phi^3 - \mathrm{e}^{-i\pi\beta} \Phi^1 \Phi^3 \Phi^2 \right) + \mathrm{h.c.}$$

the dual supergravity background was constructed by Lunin and Maldacena.

- The metric is of the form $\mathrm{AdS}_5 \times \tilde{S}^5$, where $\tilde{S}^5$ is a deformed sphere with isometry U(1)×U(1)
- All the bosonic fields ($i.e.$ the complex dilaton (τ), the $\mathrm{R}\otimes\mathrm{R}$ four-form ($C_{(4)}$) and the $\mathrm{NS}\otimes\mathrm{NS}$ and $\mathrm{R}\otimes\mathrm{R}$ two-forms ($B_{(2)}, C_{(2)}$) have a non-trivial dependence on the coordinates of the deformed sphere.

Non-supersymmetric β-deformation

The previous β-deformation can be generalised to a non-supersymmetric theory involving three deformation parameters, $\gamma_i\,,\ i=1,2,3$. This more general deformation can also be obtained introducing $*$-star products

$$\mathcal{S}=\int \mathrm{d}^4x\,\mathrm{Tr}\Big(\frac{1}{2}F^{\mu\nu}F_{\mu\nu}+D^{\mu}\bar{\varphi}_{AB}D_{\mu}\varphi^{AB}-2i\,\lambda^{A}\bar{\not{D}}\bar{\lambda}_{A}$$

$$-2\sqrt{2}g\Big([\lambda^{A},\lambda^{B}]_{*}\bar{\varphi}_{AB}+[\bar{\lambda}_{A},\bar{\lambda}_{B}]_{*}\varphi^{AB}\Big)-\frac{g^2}{2}[\varphi^{AB},\varphi^{CD}]_{*}[\bar{\varphi}_{AB},\bar{\varphi}_{CD}]_{*}\Big)$$

where for generic fields f and g:

$$[f,g]_{*}=f*g-g*f\,,\qquad f*g=\mathrm{e}^{i\pi\gamma_i\varepsilon^{ijk}\,q^f_j q^g_k}fg$$

$q^f_i\,,\ q^g_i\,,\ i=1,2,3$ denote the charges of the fields f and g with respect to a choice of the U(1)×U(1)×U(1) Cartan subalgebra of the $\mathcal{N}=4$ R-symmetry.

The charges of the various fields in the $\mathcal{N}=4$ multiplet can be chosen as follows

	q_1	q_2	q_3
φ^{14}	1	0	0
φ^{24}	0	1	0
φ^{34}	0	0	1
φ^{23}	-1	0	0
φ^{13}	0	-1	0
φ^{12}	0	0	-1

	q_1	q_2	q_3
λ^1	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
λ^2	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
λ^3	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
λ^4	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
A_μ	0	0	0

Deformations in light-cone superspace

The deformed theories have the same field content as $\mathcal{N}=4$ SYM. They can be realised in the same $\mathcal{N}=4$ light-cone superspace, even if they possess less supersymmetry, introducing a new superspace star-product.

Given $\mathcal{N}=4$ superfields, $F = F(z)$ and $G = G(z)$,

$$F = f^{(0)} + f_A^{(1)}\theta^A + f_{AB}^{(2)}\theta^A\theta^B + \cdots\,, \quad G = \cdots$$

we define

$$F \star G = F\mathrm{e}^{i\pi(\overleftarrow{Q}_{(1)}\overrightarrow{Q}_{(2)} - \overleftarrow{Q}_{(2)}\overrightarrow{Q}_{(1)})}G$$

where

$$Q_{(1)} = \textstyle\sum_{A=1}^{4} \alpha_A^{(1)} q_A\,, \qquad Q_{(2)} = \sum_{A=1}^{4} \alpha_A^{(2)} q_A$$

$$\overrightarrow{q}_A = \theta^A \overrightarrow{\frac{\partial}{\partial\theta^A}} - \bar{\theta}_A \overrightarrow{\frac{\partial}{\partial\bar{\theta}_A}}\,, \qquad \overleftarrow{q}_A = \overleftarrow{\frac{\partial}{\partial\theta^A}}\theta^A - \overleftarrow{\frac{\partial}{\partial\bar{\theta}_A}}\bar{\theta}_A$$

In order to write the deformed theories in light-cone superspace we think the U(1) charges as being associated with the fermionic coordinates in the superfield expansion instead of the component fields.

The deformed theory in then simply obtained replacing ordinary products by $\star$-products in the $\mathcal{N}=4$ light-cone superspace action.

To match the component action

$$\alpha_A^{(a)} = \alpha_A^{(a)}(\gamma)\,, \quad a = 1, 2\,, \ \ A = 1, \ldots, 4$$

β-deformed light-cone superspace action

$$\mathcal{S} = \int \mathrm{d}^4x \int \mathrm{d}^4\theta\, \mathrm{d}^4\bar\theta\ \mathrm{Tr}\Bigg\{ -\Phi\, \frac{\bar d^4 \Box}{\partial_-^4}\, \Phi$$

$$+ i\, g \left(\frac{\bar d^4}{\partial_-^3} \Phi\, [\Phi, \bar\partial \Phi]_\star + \frac{1}{48} \frac{1}{\partial_-}\, \Phi\, [\frac{\bar d^4}{\partial_-^2}\Phi, \partial \frac{\bar d^4}{\partial_-^2}\Phi]_\star \right)$$

$$+ \frac{g^2}{16} \left(\frac{1}{\partial_-} [\Phi, \partial_- \Phi]_\star \frac{1}{\partial_-} [\frac{\bar d^4}{\partial_-^2}\Phi, \partial_- \frac{\bar d^4}{\partial_-^2}\Phi]_\star + \frac{1}{2} [\Phi, \frac{\bar d^4}{\partial_-^2}\Phi]_\star\, [\Phi, \frac{\bar d^4}{\partial_-^2}\Phi]_\star \right) \Bigg\}$$

With a suitable choice of the parameters $\alpha_A^{(a)}(\gamma)$, after computing the θ integrals this action reproduces exactly the component form of the three-parameter non-supersymmetric deformation.

The special case of the $\mathcal{N}=1$ β-deformed SYM is recovered with $\alpha_4^{(1)} = \alpha_4^{(2)} = 0$.

Thanks to the properties of the $\star$-product defined above many of the calculations done for $\mathcal{N}=4$ SYM can be repeated with minimal modifications for the deformed theories.

All-order ultra-violet finiteness

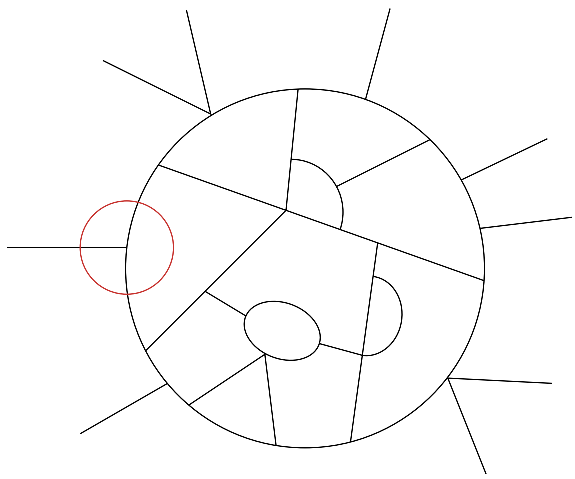
The proof of scale invariance for the deformed theories is based on Weinberg's power counting theorem. It follows the same steps as in the $\mathcal{N}=4$ case. It is possible to prove that every Green function in the theory is ultra-violet finite in the planar approximation.

Outline

1. Preliminary estimate of superficial deg. of divergence, δ: $\delta = 0$ for any superdiagram if all momenta contribute to the loop integrals.
2. Analysis of supergraphs distinguishing internal and external lines:

   manipulations of superspace expressions allow to reduce δ to a negative value for any diagram.
3. The same analysis is repeated for all subgraphs in an arbitrary supergraph.
4. The Dyson/Weinberg power counting theorem implies that all Green functions are ultra-violet finite.

For an arbitrarily complicated diagram one starts from an external leg and analyses loops sequentially moving inwards.



Generalisations

$$W = \int \mathrm{d}^4x\, \mathrm{d}^2\theta\, g\, \mathrm{Tr}\left\{ h\, \varepsilon_{IJK}\, \Phi^I [\Phi^J, \Phi^K]_* + k_I \left(\Phi^I\right)^3 \right\} + \mathrm{h.c.}$$

In general these theories are expected to be finite only if certain relations among the parameters are satisfied.
If $\gamma_i = \beta$, $i = 1, 2, 3$ and $k_I = k$, $I = 1, 2, 3$ only one relation is required $\rightarrow \gamma(g, h, \beta, k, n) = 0$

- Complex deformation parameters, γ_i

The manipulations used to prove the finiteness of the deformed theories remain valid if the parameters γ_i are complex. The light-cone superspace formulation, which does not require regularisation, may allow to extend the results to this case.
The case of complex γ_i's is interesting in connection with integrability. It may allow to shed light on the interplay of integrability, supersymmetry and conformal invariance.

- $h \neq 1$

Deformations with $h \neq 1$ can be realised in light-cone superspace

$$\mathcal{S} = c \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar{\theta}\, \left[\mathcal{L}_\star + \alpha\, \mathcal{L}_{\mathcal{N}=4} \right]$$

with $h = h(\alpha)$, $c = c(\alpha)$.

- $k_I \neq 0$

For this class of deformations a formulation in light-cone superspace may also be possible.
In this case the dual supergravity geometry is not known.

- Beyond the planar approximation

The light-cone superspace formalism allows to study systematically the effect of the $1/N$ corrections which break the scale invariance of the β-deformed theory.
The only step in the proof of finiteness which fails at finite N is the preliminary estimate of the superficial degree of divergence.
The general strategy for light-cone superspace calculations is to first project the external states onto given components, then perform the θ integrations and finally the momentum integrals. All but the last fermionic integration can be performed using δ-functions, leading to expressions which are local in θ.
The divergences in non-planar diagrams are due to momentum factors arising from the θ integrals

$$\delta^8(\theta_1-\theta_2)\left[A \star d^4\bar{d}^4\delta^8(\theta_1-\theta_2) \star B\right] = AB\delta^8(\theta_1-\theta_2)+O(k_-)$$

The structure of the superspace fermionic integrals should allow to systematically determine the condition $\beta = \beta(g, N)$ that renders the theory finite beyond the planar limit.

A world volume theory for multiple M2-branes?

Constructing a three-dimensional theory describing the world volume dynamics of a stack of coincident M2-branes has been a long-standing problem.

A simple free theory for a single M2-brane is know, but no generalisation with the properties required by multiple M2-branes has been found.

One expects a superconformal field theory with $\mathcal{N}=8$ supersymmetry in $d=3$, resulting in a OSp$(8|4)$ symmetry group. The theory should contain 8 scalars associated with the transverse directions and 8 fermions. A natural way to introduce a local U(N) symmetry is to couple a Chern–Simons gauge field.

However, a no-go theorem seems to rule out this possibility.

Recent progress:

- Basu and Harvey: a generalised Nahm equation for multiple M2-branes ending on a M5-brane
- Bagger and Lambert: a new $\mathcal{N}=8$ Chern–Simons theory – the no-go theorem is circumvented by considering fields valued in a Lie three-algebra.

Bagger–Lambert model

$$\mathcal{L} = -\frac{1}{2}D_\mu X^{aI} D^\mu X^I_a + \frac{i}{2}\bar{\Psi}^a \Gamma^\mu D_\mu \Psi_a + \frac{i}{4} f_{abcd} \bar{\Psi}^b \Gamma^{IJ} X^{cI} X^{dJd} \Psi^a$$
$$-\frac{1}{12}\left(f_{abcd} X^{aI} X^{bJ} X^{cK}\right)\left(f_{efg}{}^{d} X^{eI} X^{fJ} X^{gK}\right)$$
$$+\frac{1}{2}\,\varepsilon^{\mu\nu\lambda}\left(f_{abcd} A^{ab}_\mu \partial_\nu A^{cd}_\lambda + \frac{2}{3} f_{aef}{}^{g}\, f_{bcdg}\, A^{ab}_\mu A^{cd}_\nu A^{ef}_\lambda\right)$$

$X^I \longrightarrow$ 8 scalar fields

$\Psi \longrightarrow$ 1 d=11 Majorana spinor, satisfying the constraint

$$\Gamma_{012}\Psi = -\Psi$$

Both fields are valued in a Lie three-algebra:

$X^I = X^I_a T^a$, $\Psi = \Psi_a T^a$,

with

$$[T^a, T^b, T^c] = i f^{abc}{}_d T^d$$

$$h^{ab} = \mathrm{Tr}\left(T^a T^b\right)$$

The gauge field has two algebraic indices, A^{ab}_μ. Covariant derivative

$$D_\mu{}^a{}_b = \delta^b{}_a\, \partial_\mu - A_\mu{}^b{}_a\,, \quad \text{with} \quad A_\mu{}^b{}_a = f^{cdb}{}_a\, A_{\mu\, cd}$$

The $\mathcal{N}$=8 supersymmetry algebra closes on-shell up to a gauge transformation.

M2-branes in light-cone superspace

The light-cone gauge formulation retains only the physical propagating degrees of freedom $\rightarrow$ The Chern–Simons gauge field can be eliminated leading to a theory of 8 scalars and 8 fermions.

The B–L theory must be expressed in terms of three-dimensional spinors by a suitable choice of eleven-dimensional Γ-matrices:

$$\Gamma^\mu = \rho^9 \otimes \gamma^\mu\,, \quad \mu = 0, 1, 2$$
$$\Gamma^I = \rho^{I-2} \otimes \mathbb{1}_2\,, \quad I = 3, \ldots, 10$$

$\mathcal{P}_\pm$ projectors are constructed from the three-dimensional γ^μ's. The field content is simply 8 real 1 component spinors and 8 scalars.

The gamma-matrices can be chosen so as to give four complex three-dimensional spinors. The SO(8) R-symmetry is broken to SO(6)$\sim$SU(4).

One can then decompose the scalars as

$$X^I \rightarrow X, \bar{X}, X^i\,, \quad i = 1, \ldots, 6$$

The component fields can then be packaged in the same light-cone superfield used for $\mathcal{N}$=4 SYM:

$$\Phi \sim X + \theta^A \bar{\psi}_A + t^i_{AB} \theta^A \theta^B X_i + \varepsilon_{ABCD}\, \theta^A \theta^B \theta^C \psi^D + \theta^4 \bar{X}$$