

The effect of finite lattice spacing in the epsilon regime of QCD

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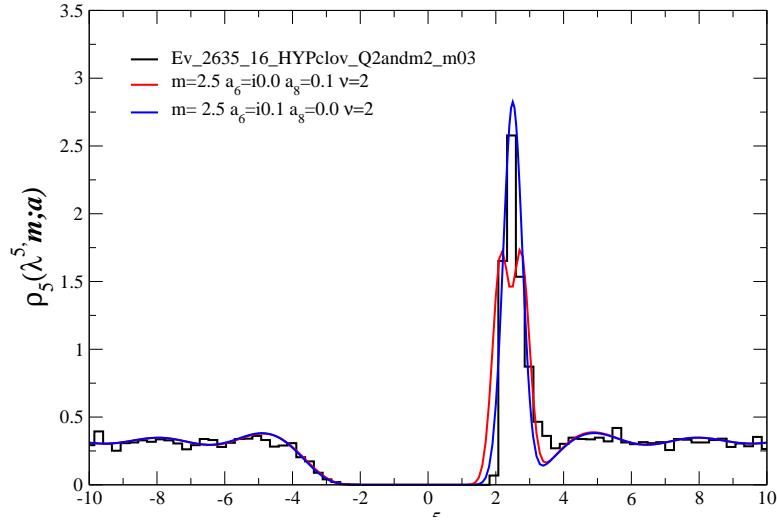
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Motivation I: lattice QCD vs. analytical results



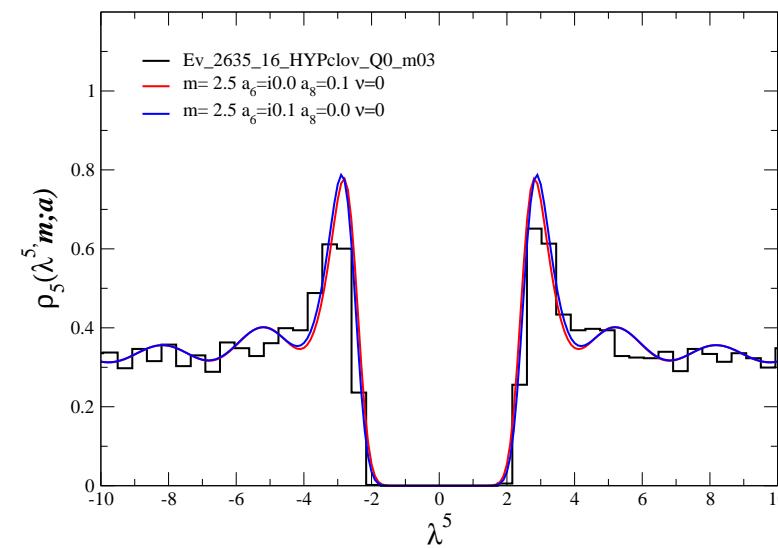
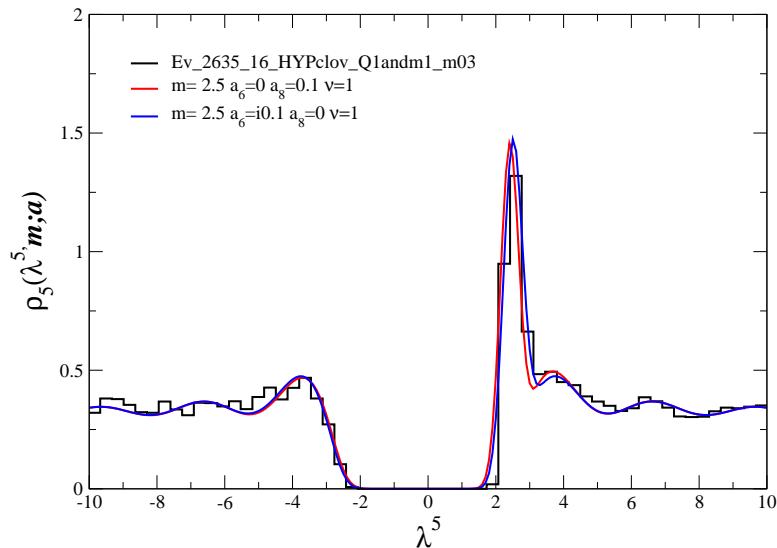
$$D_5 = \gamma_5(D_W + m))$$

quenched density of Hermitian Wilson Dirac operator ($\nu = 0, 1, 2$)

[Damgaard, Splittorff, Verbaarschot+G.A.'11]

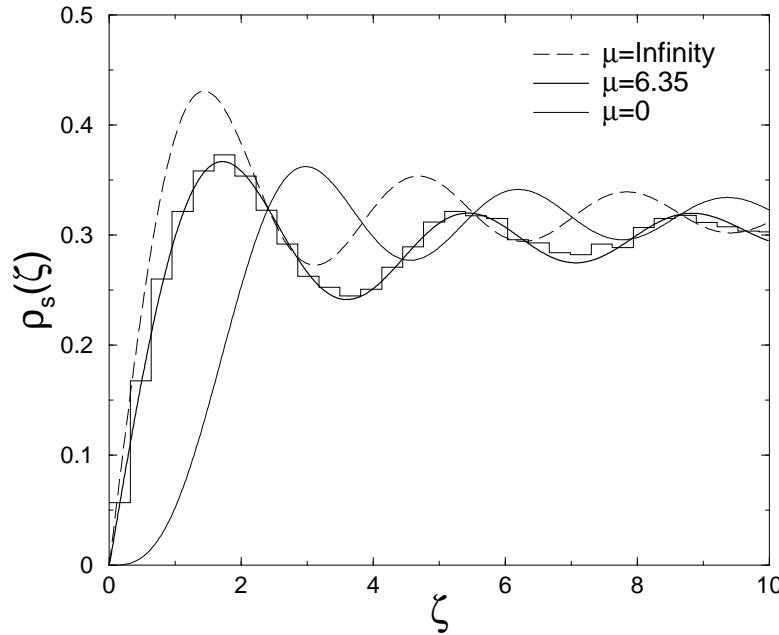
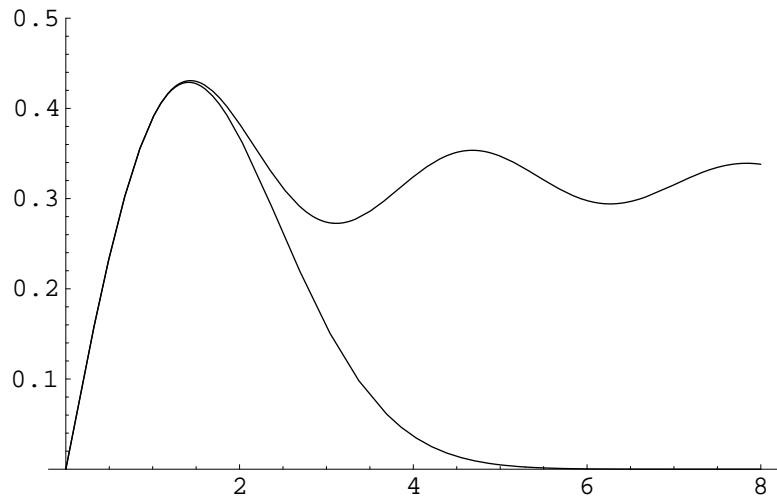
fit parameters $\Sigma, a^2 W_8, a^2 W_6$

[Damgaard, Heller, Splittorff '13]



- checked scaling with Volume V and mass m : $m\Sigma V, a^2 V W_{6,8}$

Motivation II: examples for analytic results at $a = 0$ and $a \neq 0$

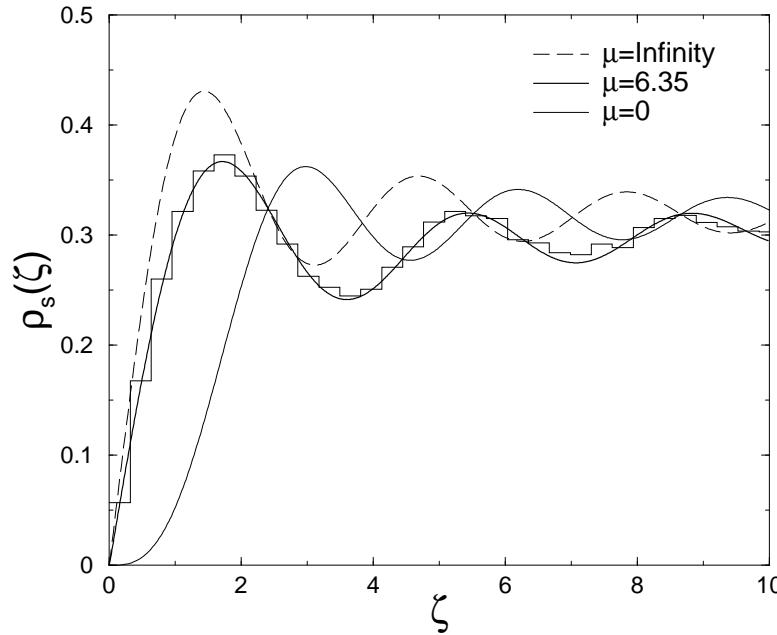
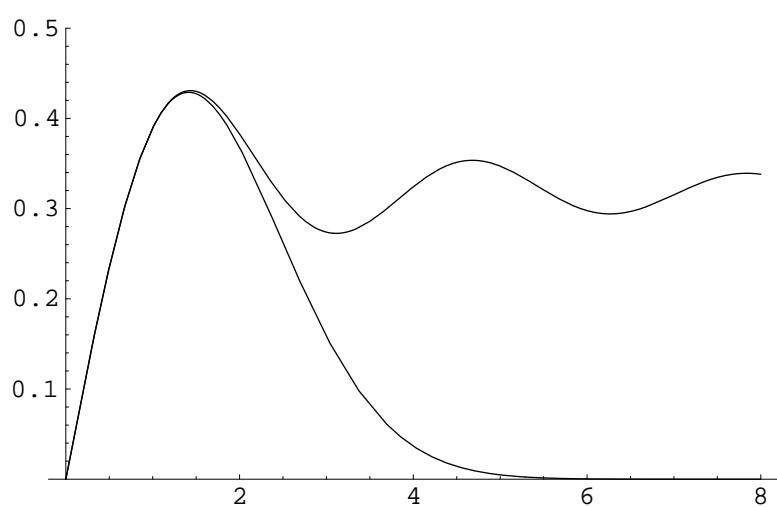


$$a = 0 \text{ left: } \boxed{\rho_S(\zeta) = \frac{\zeta}{2} (J_0(\zeta)^2 + J_1(\zeta)^2)} \quad \boxed{p_1(\zeta) = \frac{1}{2}\zeta e^{-\frac{1}{4}\zeta^2}} \quad \zeta = \lambda \Sigma V$$

quenched **QCD-D** density [SV 93] & 1st eigenvalue ($\nu = 0$) [DNW 98]

right: unquenched density [Damgaard et al. 00], $\rho_S(\zeta; \mu = m\Sigma V)$

Motivation II: examples for analytic results at $a = 0$ and $a \neq 0$



$a = 0$ left: $\boxed{\rho_S(\zeta) = \frac{\zeta}{2} (J_0(\zeta)^2 + J_1(\zeta)^2)}$ $\boxed{p_1(\zeta) = \frac{1}{2}\zeta e^{-\frac{1}{4}\zeta^2}}$ $\zeta = \lambda \Sigma V$

quenched **QCD-D** density [SV 93] & 1st eigenvalue ($\nu = 0$) [DNW 98]

right: unquenched density [Damgaard et al. 00], $\rho_S(\zeta; \mu = m\Sigma V)$

$a \neq 0$ $\rho_S(\hat{x}) = \int \frac{d\hat{y}ds dr}{\pi} e^{-\frac{(\hat{x}+\hat{y})^2(\hat{x}-\hat{y}+4i\hat{a}(r-s))}{32\hat{a}^2}} \left(\text{erf}\left(\frac{\hat{y}-\hat{x}+2\hat{m}}{4\sqrt{2\hat{a}^2}}\right) + \text{erf}\left(\frac{\hat{y}-\hat{x}-2\hat{m}}{4\sqrt{2\hat{a}^2}}\right) \right)$

$$\times e^{-s^2-r^2} \int_0^1 dt I_0(\sqrt{t(\hat{m}^2 - (\hat{y} + 4is\hat{a})^2)}) I_0(\sqrt{t(\hat{m}^2 - (\hat{x} + 4ir\hat{a})^2)})$$

Introduction I

- framework **chiral perturbation theory (chPT)**
 - = effective field theory of Goldstone bosons (GB)
 - + in the epsilon-regime of chPT = unphysical limit
 - + $\mathcal{O}(a^2)$ effects from finite lattice spacing (here: Wilson chPT)
- **Why study unphysical limit?**
 - analytic, non-perturbative results possible
 - universal regime \Leftrightarrow chiral Random Matrix Theory (chRMT)
 - very successful in studying finite-Volume V effects (at $a = 0$)
 - ▷ Dirac operator spectrum
 - ▷ effect of topology on the lattice
 - ▷ 2-point functions
 - ▷ determination of low energy constants (LEC):
 F pion decay constant, Σ chiral condensate

Introduction II

Why study $a \neq 0$ effects - non-universal?

- both finite- V and finite $-a$ corrections may become important
- the **order of limits** is important (and subtle):
 - 1) continuum limit $a \rightarrow 0$
 - 2) thermodynamic limit $V \rightarrow \infty$
 - 3) chiral limit $m \rightarrow 0$
 - otherwise may loose spontaneous chiral symmetry breaking (chSB)
 - or end up in an unphysical phase: Sharpe-Singleton (no $m = 0$ GB) vs. Aoki scenario ($m = 0$ GB at $a \neq 0$)
- determine analytically dependence on LEC $W_{6,7,8}$ in epsilon WchPT
- important: we CANNOT predict the physical values of LECs as functions of N_f , N_c , α_S etc. → determined by fit to lattice **QCD**

Outline

- Part I:
QCD, setup, symmetries of Dirac operator (cont. & $a \neq 0$), chSB
- Part II:
chPT, epsilon- vs. p-regime, Wilson chPT, map to (W)chRMT and 1-loop corrections
 - applications: constraints on sign of LEC, Aoki phase
- Part III:
spectral correlations and chRMT, Banks-Casher relation, generating function from chPT or chRMT,
sketch of solution of RMT, individual eigenvalue distributions
- Concluding remarks

Part I: Quantum Chromodynamics (QCD)

Symmetries of the continuum Euclidean QCD Dirac operator

- **chiral symmetry** in continuum $a = 0$

$$\{\not{D}, \gamma_5\} = \not{D}\gamma_5 + \gamma_5\not{D} = 0 \quad \text{with } \gamma_5 \equiv \begin{pmatrix} \mathbf{1}_N & 0 \\ 0 & -\mathbf{1}_{N+\nu} \end{pmatrix}$$

- probe: breaking by quark mass $m \neq 0$
- soft: $\lim m \rightarrow 0, V \rightarrow \infty$ such that $mV \sim \text{const.}$
- eigenvalues: **trivial shift** $\not{D} + m\mathbf{1}$
- QCD-Dirac operator \not{D} in chiral basis
 - anti-Hermitian $\not{D} = -\not{D}^\dagger \equiv \begin{pmatrix} 0_N & W \\ -W^\dagger & 0_{N+\nu} \end{pmatrix}$ ($2N + \nu \sim \text{Volume}$)
 - eigenvalue pairs $\pm i\lambda_j, \lambda_j \in \mathbb{R} > 0$ & ν **zero eigenvalues**
- vs. chRMT **chiral Gaussian Unitary Ensemble (chGUE)**

$$\mathcal{Z}_\nu = \int dW \det[\not{D} + m\mathbf{1}]^{N_f} \exp[-N \Sigma \text{Tr}WW^\dagger] \quad [\text{Shuryak, Verbaarschot 93}]$$

W complex $N \times (N + \nu)$ matrix, N_f flavours of mass m

Setup QCD

- full theory (Euclidean) $+\varepsilon^{\mu\nu\rho\sigma}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}$ topological term

$$\mathcal{Z}_{\text{QCD}} \equiv \int [dA][d\Psi] \exp[-\text{Tr} \int \bar{\Psi}(\not{D} + M)\Psi + \mathcal{FF} + i\Theta\mathcal{F}\tilde{\mathcal{F}}]$$

with N_f quarks Ψ of masses M , gauge fields A and field strength \mathcal{F}

- integrate out quarks

$$\mathcal{Z}_{\text{QCD}} \equiv \sum_{\nu} \int [dA]_{\nu} \prod_{f=1}^{N_f} \det[D + m_f] \exp[-\text{Tr} \int \mathcal{F}^2 + i\Theta\nu]$$

- **properties of the Dirac operator:**

- index theorem: ν # zero modes \longleftrightarrow winding # of A

- sum over topology $\mathcal{Z}_{\text{QCD}} \equiv \sum_{\nu} \mathcal{Z}_{\nu} \exp[i\Theta\nu]$

- diagonalise: $D\psi_k = i\lambda_k\psi_k$ in finite V , Euclidean $D^\dagger = -D$

- $\{D, \gamma_5\} = 0$: eigenvalues $\neq 0$ in pairs $\pm i\lambda_k$

- formally

$$\mathcal{Z}_{\text{QCD}} = \sum_{\nu} \int_0^{\infty} [d\lambda] \prod_{f=1}^{N_f} m_f^{\nu} e^{i\nu\Theta/N_f} \prod_k (\lambda_k^2 + m_f^2) \exp[-\text{Tr} \int \mathcal{F}^2[\lambda]]$$

Symmetries of the discretised Dirac operator: Wilson

- Lattice QCD with **finite spacing**: $\mathbb{D} \rightarrow \mathbb{D}(a \neq 0)$ not unique
 - problem: doublers, cannot keep all continuous symmetries

- Wilson: $\boxed{\mathbb{D} \rightarrow \mathbb{D} + a\Delta \equiv D_W}$ add Laplacian (Hermitian):
 - removes doubles, breaks chiral symmetry

$$\boxed{\{D_W, \gamma_5\} = \mathcal{O}(a) \neq 0 \text{ and } D_W \neq -D_W^\dagger}$$

- **residual symmetry for $a \neq 0$:**

$$\boxed{D_W^\dagger = \gamma_5 D_W \gamma_5 \Rightarrow D_W = \begin{pmatrix} aA & W \\ -W^\dagger & aB \end{pmatrix}}$$

$\det[D_W - \lambda] \in \mathbb{R}$: spectrum or real or complex conj.

- **Hermitian Wilson Dirac operator:** $(\gamma_5 D_W)^\dagger = D_W^\dagger \gamma_5 = \gamma_5 D_W$

$$\boxed{D_5 \equiv \gamma_5(D_W + m\mathbf{1}) = \begin{pmatrix} aA+m & W \\ W^\dagger & -aB-m \end{pmatrix}} \rightarrow \text{random matrix}$$

- spectrum of D_5 remains real, shift in m non-trivial

Chiral Symmetry Breaking (XSB)

- explicit breaking:

mass term $\bar{\Psi}_R M \Psi_L + \bar{\Psi}_L M \Psi_R$, trafo $\Psi_{L,R} \rightarrow U_{L,R} \Psi_{L,R}$

invariant if $U_L = U_R$ & $M = m\mathbf{1}$
$$U_L(N_f) \times U_R(N_f) \rightarrow U(N_f)$$

- spontaneous breaking:

chiral condensate $\Sigma \equiv |\langle \bar{\Psi} \Psi \rangle| = |\langle \bar{\Psi}_R \Psi_L + \bar{\Psi}_L \Psi_R \rangle| \equiv \Sigma \sim \rho_D(0) \neq 0$

- non-perturbative phenomenon (breaks as mass term $M = m\mathbf{1}$)

- in **QCD**: $U_A(1)$ explicitly broken by anomaly, $U_V(1)$ unbroken

$$SU_L(N_f) \times SU_R(N_f) \rightarrow SU_V(N_f)$$
 QCD XSB pattern

- **Goldstone Theorem**: massless Goldstone Bosons

→ low energy effective theory = **chiral Perturbation Theory (XPT)**

- \exists further simplification: **epsilon-regime** of XPT

\Leftrightarrow **Random Matrix Theory (RMT)**

Part II: Chiral Perturbation Theory (XPT)

Standard chiral perturbation theory

Integrating out all non-Goldstone modes in QCD:

- in a box $V = L^4$: valid for momenta $1/L \ll \Lambda$ non-Goldstone scale

$$\mathcal{Z}_{\chi PT} \equiv \int_{SU(N_f)} [dU(x)] \exp[-\int dx \text{Tr } \mathcal{L}(U, \partial U)]$$

- expansion in $U(x) = U_0 \exp[i\pi_k(x)\sigma_k/F]$: $\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \dots$

$$\mathcal{L}_2 = \frac{1}{4} F^2 \partial_j U(x) \partial_j U^\dagger(x) + \frac{1}{2} \Sigma M (U(x) + U^\dagger(x))$$

- LEC: **Pion decay constant F** & **chiral condensate Σ**

$M = \text{diag}(m_u, m_d, \dots)$ **quark masses**, $m_\pi^2 F^2 = N_f \Sigma m$ [GOR]

- inverse Fourier trafo: $\int d\Theta e^{i\nu\Theta} \int_{\mathbf{SU}(N_f)} dU = \int_{\mathbf{U}(N_f)} dU \det[U]^\nu$

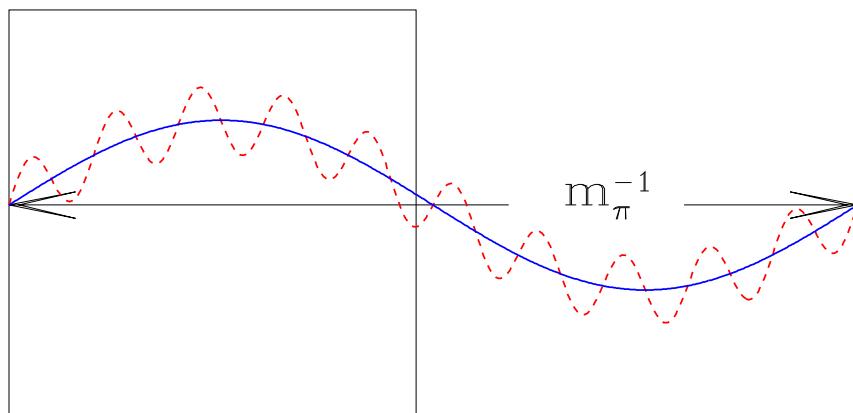
→ project to sectors of fixed topology $\mathcal{Z} = \sum_\nu e^{-i\nu\Theta} \mathcal{Z}_\nu$

- $a \neq 0$: more terms in \mathcal{L}_2 , with more LEC

The ε -regime in XPT

- standard p -regime counting: $\boxed{\partial \sim m_\pi \sim 1/L}$
- ε -regime counting [Gasser, Leutwyler 87]
$$\boxed{m_\pi \sim \frac{1}{L^2} \ll \frac{1}{L} = \varepsilon = V^{-4}} \Rightarrow ((p^2 + m_\pi^2)V)^{-1} = \mathcal{O}(1) \text{ for } p = 0$$

fluctuations of quadratic term large



unphysical regime !!

fig: [Bietenholz et al. physics/0309072]

The ε -regime in XPT: zero-mode dominance

- zero-modes U_0 dominate: non-perturbative treatment
→ \mathcal{L}_2 exact & factorisation:

$$\mathcal{Z}_{\nu, \varepsilon \chi \text{PT}} = \int_{U(N_f)} dU_0 \det[U_0]^\nu e^{-\frac{1}{2} \Sigma V \text{Tr}(\textcolor{blue}{M}(U_0 + U_0^\dagger))} \times \int [d\pi] e^{-\int dx \frac{1}{2} (\partial \pi)^2}$$

- resulting group integral:

$$\mathcal{Z}_{\nu, \varepsilon \chi \text{PT}} \sim \det_{1, \dots, N_f} \left[\hat{m}_k^{j-1} I_{\nu+j-1}(\hat{m}_k) \right] / \Delta_{N_f}(\hat{m}_f^2) \times \text{const.}$$

[Brower,Tan, Rossi 81, etc], with $\hat{m} = mV\Sigma$,

$$\Delta(m^2) = \prod_{f>g}^{N_f} (m_f^2 - m_g^2) \text{ Vandermonde det}$$

- example $N_f = 2$: $\mathcal{Z}_\nu = I_\nu(\hat{m})^2 - I_{\nu+1}(\hat{m})I_{\nu-1}(\hat{m})$

same result from RMT when $\lim_{N \rightarrow \infty} mN = \hat{m}$

What can we learn from $\varepsilon\chi\text{PT}$?

- LO: non-zero modes = Gaussian free fields \sim overall constant
- study quark mass dependence of partition function:
 - Taylor expand in m_q formal expression for \mathcal{Z}_{QCD} in terms of λ_k

$$\mathcal{Z}_\nu(\{m_q\}) = \int_0^\infty [d\lambda] \prod_{f=1}^{N_f} m_f^\nu \prod_k (\lambda_k^2 + m_f^2) \exp[-\text{Tr} \int \mathcal{F}^2[\lambda]]$$

- compare to group integral & expand Bessel functions
- Leutwyler-Smilga sum rules:

$$m^\nu \left(1 + m^2 \left\langle \sum_k \frac{1}{\lambda_k^2} \right\rangle_{\text{QCD}} + \dots \right) = I_\nu(m) \Leftrightarrow$$

$$\left\langle \sum_k \frac{1}{V \sum \lambda_k^2} \right\rangle_{\text{QCD}} = \frac{1}{4(\nu+1)}$$

- computation of spectral density possible \Rightarrow all sum-rules

Limitations of RMT and corrections from $\varepsilon\chi$ PT

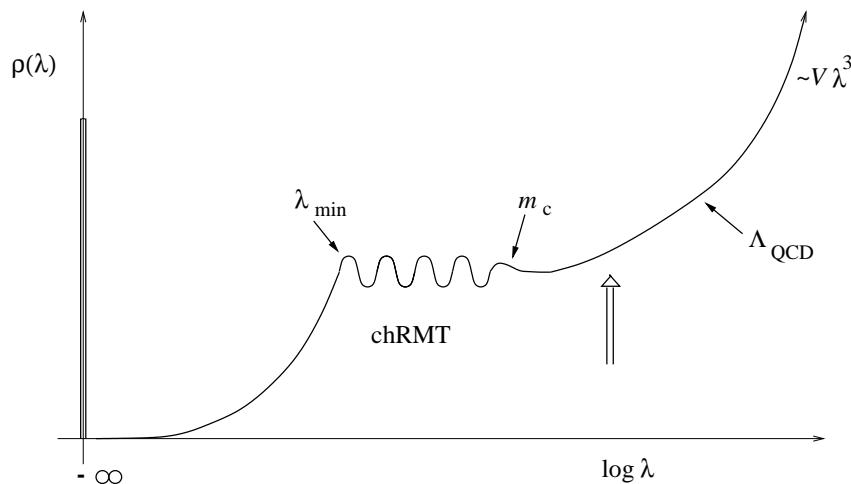


fig: [Verbaarschot]

- breakdown at Thouless Energy $\boxed{\lambda_c \sim F^2 / \sqrt{V}}$ [Osborn, Verbaarschot 01]
- 1-loop $\varepsilon\chi$ PT corrections $\Sigma_{eff} = \Sigma \left(1 - \frac{N_f^2 - 1}{N_f F} \bar{\Delta}(0) \right)$, $F_{eff} = \dots$
in $1/\sqrt{V}$ [Gasser, Leutwyler 87; Damgaard, Fukaya, DeGrand 07; G.A., Basile, Lellouch 08]
- still \Leftrightarrow RMT (at $a = 0$) with eff. LEC [Lehner, Wettig 09]

Lattice spacing $a \neq 0$: Wilson Chiral Perturbation Theory

- **Symanzik's O(a) improvement:**

close to continuum discrete lattice field theory described by effective, *continuum* field theory $S_{eff} = S_0 + aS_1 + a^2S_2 + \dots$

- here: ε Wilson chiral perturbation theory (WXPT)

$$\mathcal{Z}_\nu \sim \int_{U(N_f)} dU_0 \det[U_0]^\nu \exp \left[\frac{1}{2} V \Sigma \text{Tr} \left\{ M(U_0 + U_0^\dagger) + Z(U_0 - U_0^\dagger) \right\} \right] \times$$

$$\exp \left[-a^2 V W_8 \text{Tr}[U_0^2 + U_0^{\dagger 2}] - a^2 V W_6 \left(\text{Tr}[U_0 + U_0^\dagger] \right)^2 - a^2 V W_7 \left(\text{Tr}[U_0 - U_0^\dagger] \right)^2 \right]$$

[Sharpe, Singleton 98; Rupak, Shores 02 + Bär 04]

- **new LEC** W_j & **new scaling** $a^2 V = \mathcal{O}(1)$, (\exists other: GSM(*))
- $SU(2)$: no W_7 , only combination $c_2 = W_6 + W_8/2$
- ε WXPT \Leftrightarrow Wilson RMT at LO [Damgaard, Splittorff, Verbaarschot+G.A. 10]

Standard Map RMT to ε XPT

- $$\mathcal{Z}_\nu = \int dW \prod_f^{N_f} \det[WW^\dagger + m_f^2] e^{-N\Sigma^2 \text{Tr } WW^\dagger} \quad a = 0$$
- step 1. **Grassmann** $\det[\dots] = \int d\psi \exp[\bar{\psi} \dots \psi] \rightarrow \text{do Gauß' } \int dW$
- step 2. $\psi^4 \rightarrow \text{Hubbard-Stratonovich}$: extra $\int dQ_{N_f \times N_f} \rightarrow \text{do } \int d\psi$:

$$\mathcal{Z}_\nu \sim \int dQ \det[Q^\dagger]^\nu \det [Q^\dagger Q]^N e^{-N\Sigma^2 \text{Tr}(Q^\dagger Q - \textcolor{blue}{M}(Q+Q^\dagger))}$$
- step 3. **Saddle Point** $N \rightarrow \infty$: $Q = UR$, $R \sim 1$, + scale masses

$$\mathcal{Z}_\nu \rightarrow \int dU_0 \det[U_0^\dagger]^\nu e^{-N\Sigma \text{Tr} \textcolor{blue}{M}(U_0 + U_0^\dagger)} = \varepsilon \text{XPT}$$
- $a \neq 0$: W_8 -term extra Gauss matrices A, B [DSV 10],

$$D_W = \begin{pmatrix} aA & W \\ -W^\dagger & aB \end{pmatrix}, \text{ masses } M \rightarrow \text{diag}[(m + iz)\mathbf{1}, (m - iz)\mathbf{1}]$$

Example $a \neq 0$ eWchPT Group Integral

- for simplicity
 - consider equal mass m for all N_f flavours
 - set $W_{6,7} = 0$

\Rightarrow epsilon WchPT depends on

$$\hat{m} = m\Sigma V, \hat{a} = a^2 W_8 V$$

- 1 flavour

$$\mathcal{Z}_\nu^{(N_f=1)}(\hat{m}, \hat{a}) \sim \int_{-\infty}^{\infty} \frac{dx}{\sqrt{\pi}} e^{-x^2} \left(\frac{\hat{m} - 4ix\hat{a}}{\hat{m} + 4ix\hat{a}} \right)^{\nu/2} I_\nu \left(\sqrt{\hat{m}^2 - (4ix\hat{a})^2} \right)$$

- N_f flavours

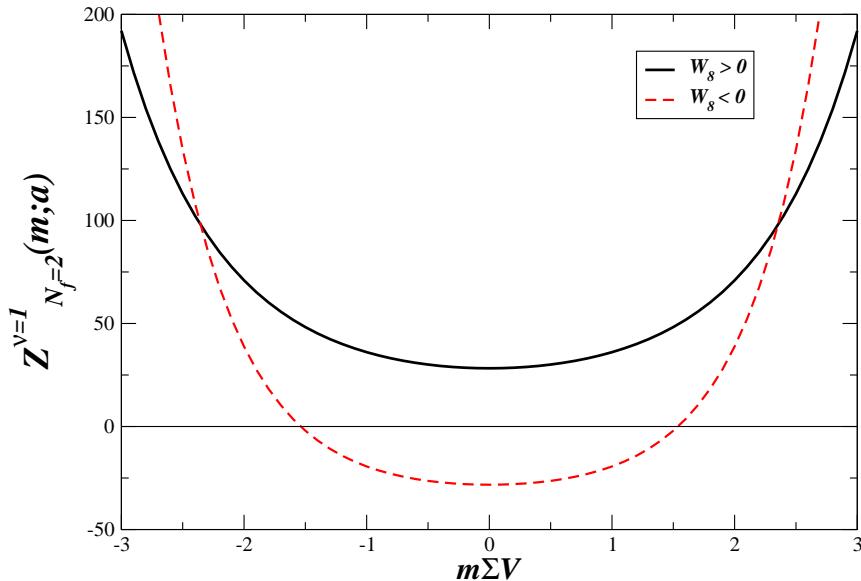
$$\mathcal{Z}_\nu^{(N_f)}(\hat{m}, \hat{a}) \sim \det_{k,j} \left[\mathcal{Z}_{\nu+k-j}^{(N_f=1)}(\hat{m}, \hat{a}) \right]$$

Toepliz determinant

- $W_{6,7} \neq 0$: extra Gaussian integrals

[G.A., Damgaard, Splittorff, Verbaarschot 10]

Positivity of partition function: constraint on LEC



- for ν odd: $\mathcal{Z}_\nu^{(N_f=2)}$ positive definite only for $W_8 > 0$ [ADSV 10]
- [M. Kieburg, K. Splittorff, J.J.M. Verbaarschot, 12]:
 - constraint $W_6, W_7 < 0$: from Hermiticity of generating function
 - $c_2 = W_6 + W_8/2$ **can have both signs**: determines Aoki or Sharpe-Singleton (only unquenched)
- Note: topology for $\nu \neq 0$ no longer well defined \rightarrow spectral flow

Lesson for the Aoki phase at NLO: SU(2)

- mean field analysis of static potential [SS 98] LO, [G.A., Pucci 12] NLO

$$\boxed{\Phi = -\frac{m}{2}\Sigma_{eff}V\text{Tr}[U + U^\dagger] + a^2c_{2,eff}V(\text{Tr}[U + U^\dagger])^2}$$

- sign and size of LEC_{eff} determine min & possible phase transition

- parametrize $\boxed{U = A\mathbf{1} + i\vec{B}\vec{\sigma}}$ $\Rightarrow \text{Tr}[U + U^\dagger] = 4A$

- potential minimum: non-trivial for $c_2 > 0$

- I) $|A_0| = 1 \Leftrightarrow U_0 \sim \mathbf{1}$ is $SU(2)$ invariant

- II) $|A_0| < 1 \Leftrightarrow |\vec{B}_0| > 0$ breaks $SU(2) \rightarrow U(1)$ **Aoki phase**

this can only happen for $\boxed{c_{2,eff} > 0}$ and $\boxed{\left|\frac{m\Sigma_{eff}}{16a^2c_{2,eff}}\right| < 1}$

\Rightarrow take continuum limit s.th. $m\Sigma_{eff} > 16a^2c_{2,eff}$ to avoid Aoki

- **NLO effects may become important for the phase boundary**

Example two-point function

$$\begin{aligned} \langle S_0(x)S_0(0) \rangle &= \frac{(\Sigma^{eff})^2}{4} \left\langle \left(\text{Tr}[U_0 + U_0^\dagger] \right)^2 \right\rangle_{\text{NLO}} \\ &\quad - \frac{\Sigma^2}{2F^2} \left\langle \text{Tr} \left[U_0^2 + U_0^{\dagger 2} \right] - 4 \right\rangle_{\text{LO}} \Delta(x) \\ &\quad - \frac{\hat{a}\Sigma c_3}{\sqrt{V}} \left\langle \left(\text{Tr}[U_0 + U_0^\dagger] \right)^3 \right\rangle_{\text{LO}}, \end{aligned} \tag{1}$$

- scalar-scalar 2-point correlation function [G.A., Pucci 12]
 Σ^{eff} depends on the LECs from LO and NLO

Part III:

Spectral Correlations and Random Matrix Theory

Relating Chiral Symmetry & Dirac Spectrum

- **Banks Casher Relation** $\boxed{\rho_D(\lambda = 0) = \frac{1}{\pi} V \Sigma}$ **macroscopic density**

$$\Sigma(\textcolor{magenta}{m}) \equiv \frac{1}{V} \partial_{\textcolor{magenta}{m}} \log \mathcal{Z}_{\text{QCD}} = \left\langle \sum_k \frac{2m}{V(\lambda_k^2 + m^2)} \right\rangle = \int d\lambda \rho_D(\lambda; \textcolor{magenta}{m}) \frac{2\textcolor{magenta}{m}}{(\lambda^2 + \textcolor{magenta}{m}^2)}$$

- $V \rightarrow \infty$ taken, then chiral $\lim_{m \rightarrow 0} \frac{2\textcolor{magenta}{m}}{(\lambda^2 + \textcolor{magenta}{m}^2)} \rightarrow \delta^{(1)}(\lambda)$

- **small eigenvalues responsible for** $\Sigma \neq 0$
(NOT true for chemical potential $\mu \neq 0$: quenched $\Sigma = 0$!,
sign problem & oscillations give $\Sigma \neq 0$) [Osborn, Splittorff, Verbaarschot]
- **spacing close to** $\lambda = 0$: ($V = L^4$)

- effect of strong coupling $\boxed{\lambda \sim \frac{1}{V} \neq \frac{1}{L}}$ free momenta

- **microscopic scaling:** $\boxed{\lim_{V \rightarrow \infty} \frac{1}{\Sigma V} \rho_D(\lambda \Sigma V)}$ **zoom into origin**

Resolvent - spectrum relation

- **Resolvent definition**

$$\Sigma(z) \equiv \langle \text{Tr} \frac{1}{D - z} \rangle \longrightarrow \int d\lambda \rho_D(\lambda) \frac{1}{i\lambda - z}$$

- **inversion** of this relation:

$$\boxed{\rho(\lambda) \sim \text{Im}[\Sigma(i\lambda)]} \text{ discontinuity}$$

ONLY if $\rho_D(\lambda)$ is INDEPENDENT of z

- how to generate resolvent: add **auxiliary quarks**

$$\boxed{\partial_m \det[D + m] = \text{Tr} \frac{1}{D-m} \det[\not{D} + m]} \text{ (from } \det[A] = \exp[\text{Tr} \log[A]]\text{:)}$$

- have to remove additional $\det[\not{D} + m]$
- alternative: Replicas $\lim_{n \rightarrow 0} \frac{1}{n} \partial_m \det[D + m]^n$

How to compute densities from $\varepsilon\chi\text{PT}$

- auxiliary fermion- boson pair $\Sigma(m) = \partial_m \left\langle \frac{\det[\not{D}+m]}{\det[\not{D}+m_B]} \Big|_{m=m_B} \right\rangle$
- e.g. density: **compute partially quenched $\varepsilon\chi\text{PT}$**
supergroup integral $U(N_f + 1|1)$ [Damgaard, Osborn, Toublan, Verbaarschot 98]
- **generating functions** for all k -point densities:
 $U(N_f + k|k)$ –supergroup integral of $\varepsilon\chi\text{PT}$:
take derivatives & discontinuities and compare to RMT
generating functions agree, so also all individual eigenvalues do!
[F. Basile, G.A. 07]

proof: **Superbosonisation Theorem** also [Littelmann, Sommers, Zirnbauer 07]

$$\int d\Psi f(\Sigma_k \Psi_k \times \Psi_k) \sim \int_{U(N_f+k|k)} dU_0 \text{ sdet}[U_0]^N f(U_0)$$

RMT eigenvalue representation

$$\mathcal{Z}_{\nu, RMT} \equiv \int dW \prod_{f=1}^{N_f} \det \begin{bmatrix} m_f & iW \\ iW^\dagger & m_f \end{bmatrix} e^{-N \text{Tr} W_j W_j^\dagger}$$

- diagonalise WW^\dagger positive definite matrix : eigenvalues $\lambda_k \geq 0$
(or singular values of W : y_k with $y_k^2 = \lambda_k$ Dirac eigenvalues)
- $\mathcal{Z}_{\nu, RMT} \sim \int_0^\infty \prod_k^N dy_k y_k^{2\nu+1} e^{-N\Sigma^2 y_k^2} \prod_f^{N_f} (y_k^2 + m_f^2) |\Delta_N(y^2)|^2$
- integrand $\equiv \mathcal{P}_{jpdf}$ joint probability distribution
- Vandermonde determinant

$$\Delta_N(\lambda) = \prod_{k>l} (\lambda_k - \lambda_l) = \exp[2 \sum_{k>l} \log[\lambda_k - \lambda_l]]$$
eigenvalues strongly coupled: log-gas, repulsion of eigenvalues

Main Result All Density Correlations

- k -point density $\rho_k \sim \int dy_{k+1} \dots dy_N \mathcal{P}_{jpdf}$
- Dyson's Theorem does $N - k$ integrations in k -point function

$$\rho_k(y_1, \dots, y_k) = \det_{1, \dots, k}[K_N(y_j, y_l)]$$

$k \times k$ determinant - only large- N kernel $K_N(y_j, y_l)$

- **example** $N_f = 0$:

$$\rho_1(y) = K_N(y, y) = \lambda^\nu e^{-N\Sigma^2 y^2} \sum_{j=0}^{N-1} L_j^\nu(y^2)^2 / h_j$$

orthogonal polynomials: Laguerre

- **Christoffel-Darboux formula** large- N limit easy:

$$K_N(x, y) = \sum_{j=0}^N P_j(x) P_j(y) / h_j \sim \frac{P_N(x) P_{N-1}(y) - P_N(y) P_{N-1}(x)}{x - y}$$

- **Bessel-asymptotic** of Laguerre:

$$\lim_{N \rightarrow \infty} \rho_S(x = \lambda N \Sigma) \sim \frac{x}{2} \left(J_\nu(x)^2 - J_{\nu-1}(x) J_{\nu+1}(x) \right)$$

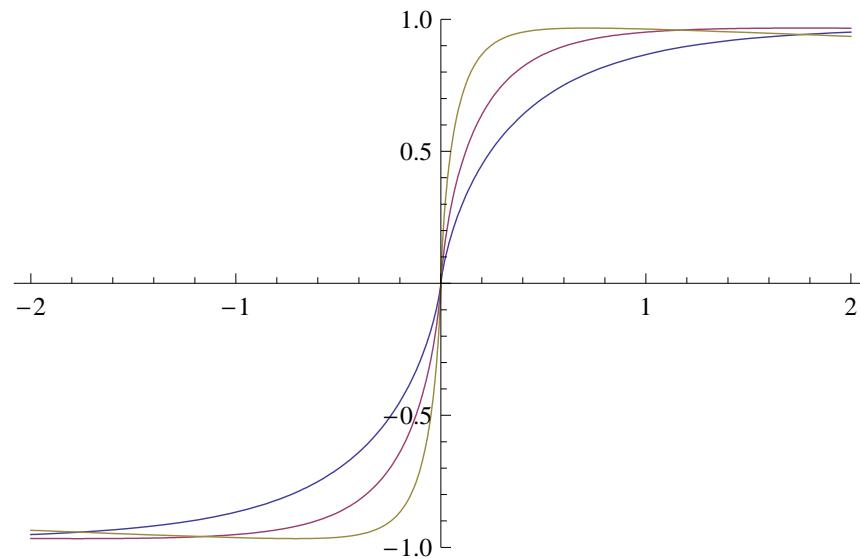
Applications

- sum rules from density, e.g. quenched

$$\left\langle \sum_k \frac{1}{V \Sigma \lambda_k^2} \right\rangle_{\text{QCD}} \rightarrow \int_0^\infty dx \frac{1}{x^2} \frac{x}{2} \left(J_\nu(x)^2 - J_{\nu-1}(x) J_{\nu+1}(x) \right) = \frac{1}{4(\nu+1)}$$

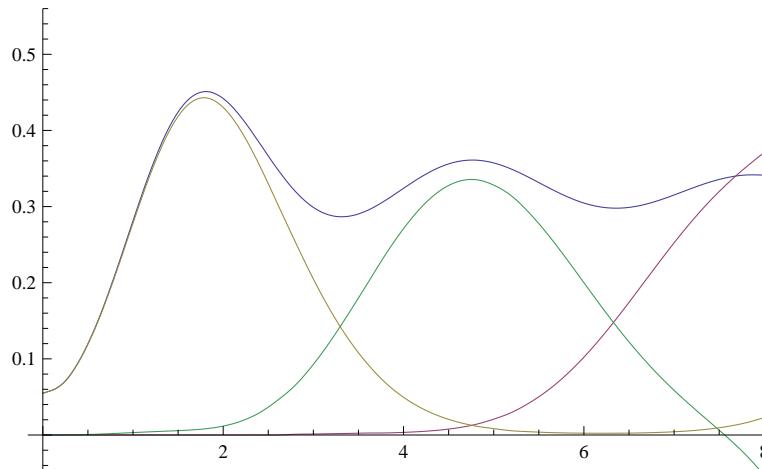
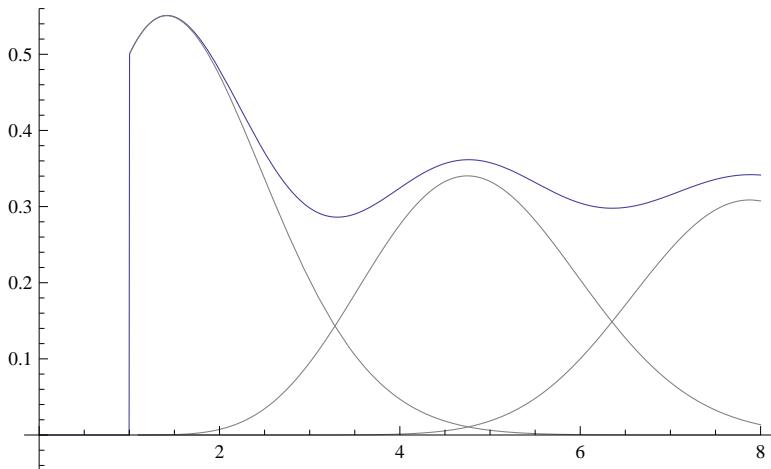
- Resolvent from density, e.g. quenched

$$\Sigma(m) = \int_0^\infty dx \frac{2m}{m^2+x^2} \rho_S(x)$$

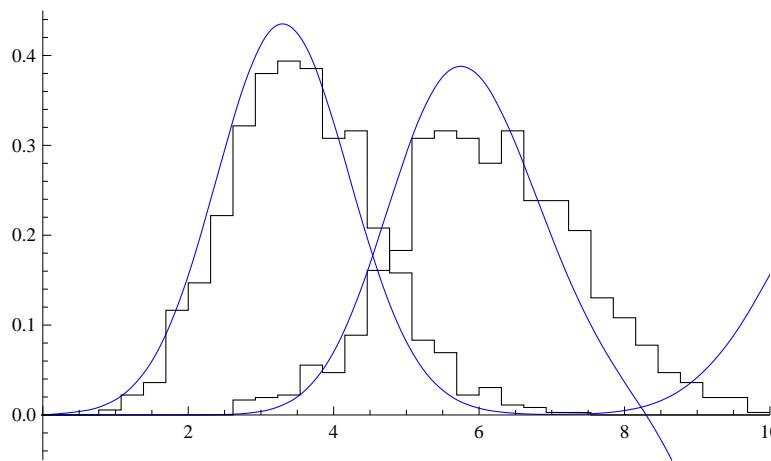
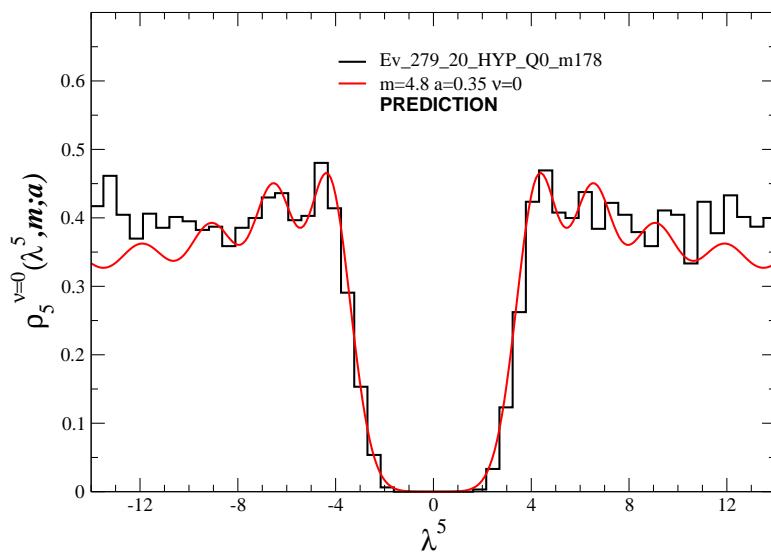


- XSB in the epsilon regime in the thermodynamical limit $V \rightarrow \infty$
- makes contact to the p -regime

RMT: distribution of individual eigenvalues at $a \neq 0$



- density and approx. indiv. eigenvalues of D_5 left: $a = 0$, right $a > 0$



- left e.g. [Damgaard, Heller, Splittorff 11], right [G.A., Ipsen 12]

Concluding remarks

- We now understand both the effect of finite volume and of finite lattice spacing
 - on the spectrum of the Dirac operator
 - LECs
 - two-point functions
- on the way we found some interesting new mathematical features of the underlying RMT
- it remains to be done
 - check unquenched predictions from Lattice: density, 2-point functions, sign of the LECs
 - 1-loop XPT effect on spectral density
- the staggered formulation has the disadvantage (so far) that no analytic solution is known of the corresponding RMT