Higher-order QCD calculations via local subtraction

Gábor Somogyi

CERN

LNF, Frascati, June 27th 2011

Why precision?

DESY 12-031, DO-TH-12/07, SFB/CPP-12-20, LPN 12–047 arxiv:1205.4991

Why Precision?

Johannes Blümlein*

Deutsches Elektronen-Synchrotron, DESY, Zeuthen, Platanenalle 6, D-15735 Zeuthen, Germany.

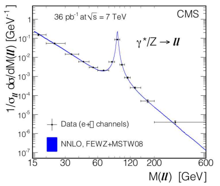
1. Introduction

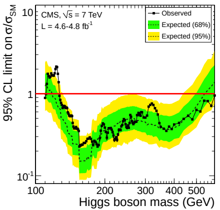
Precision matters. Any progress in the exact sciences relies both on precise measurements and highly accurate theoretical calculations. Many of the fundamental laws of physics had unavoidably to be found whenever precise data were described by theoretical concepts, often within a new framework of relations. The Rudolphine Tables of the late Tycho Brahe [1] led J. Kepler to derive his laws [2] and later I. Newton the law of gravity [3]. A. Michelson's experiments [4] led A. Einstein to Special Relativity [5], with numerous experimental confirmations in flat space-time. [1] The

Differential NNLO

A young and promising field in the LHC era

- less than a decade old
- starting from simple decays and single production processes
- moved to/moving towards complicated jet production and pair production processes at colliders
- already a significant impact on phenomenology at collider experiments





$$\alpha_s = 0.1175 \pm 0.0020\ (\text{exp}) \pm 0.0015\ (\text{theo})$$

Outline

Introduction

Basics of subtraction

Local subtraction at NNLO

The tedious part: integrating the counterterms

Integrated approximate cross sections

Outlook

Introduction

Introduction

Accurate predictions in perturbative QCD require higher-order calculations

LO QCD predictions give only order of magnitude estimates for rates and rough estimates for shapes of distributions

- large dependence on unphysical scale choices
- jets $\neq$ partons: jet structure appears only beyond LO

NLO corrections are required to obtain more realistic estimates of cross sections and better pictures of relevant distributions

Introduction

Starting from NLO, various singularities appear at intermediate stages of computation, even thought the final result is finite

- at the amplitude level, the one-loop amplitudes contain both UV and IR singularities
- at the cross section level, the virtual (loop) and real emission corrections both contain IR singularities

However, we have accumulated three decades of experience with NLO computations, and general methods exist to handle singularities

- UV singularities affect only virtual corrections and are removed by renormalization
- IR (soft and collinear) singularities cancel between virtual and real corrections for properly defined ("IR safe") observables. Several methods are known how to make this cancellation explicit.

Introduction

At NLO general methods exist to handle and cancel IR singularities

Frixione, Kunszt, Signer (1995)
Catani, Seymour (1996)

The bottleneck for many years has been computing the relevant one-loop amplitudes

Massive progress in last few years

- powerful new methods based on unitarity, recursion relations
- new general tools have been/are being developed to compute one-loop amplitudes: GoSam, Helac-NLO, MadLoop. . .

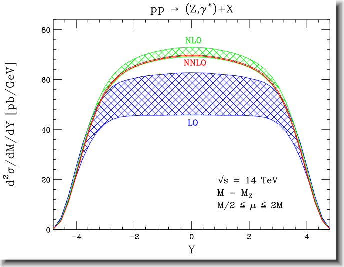
Fair to say that the problem of computing NLO corrections is essentially solved for general processes

Can we go to NNLO?

Do we need NNLO?

In certain cases precision QCD requires computations beyond NLO

- NLO corrections are large:
  - Higgs production from gluon fusion in hadron collisions
- for benchmark processes measured with high experimental accuracy:
  - α_s measurements form e^+e^- event shapes
  - W, Z production
  - heavy quark hadroproduction
- reliable error estimate is needed:
  - processes relevant for PDF determination
  - important background processes



(Anastasiou, Dixon, Melnikov, Petriello, Phys. Rev. **D69** (2004) 094008.)

In short, we need NNLO when NLO fails to do its job

Status

Processes measured to few precent accuracy

- $e^+e^- \to 3j$
- $ep \to (2+1)j$
- $pp \to j + X$
- $pp \to V$
- $pp \to V + j$
- $pp \to t\bar{t}$

Processes with potentially large radiative corrections

- $pp \to H$
- $pp \to H + j$
- $pp \to VH$
- $pp \to VV$

Status

Processes measured to few precent accuracy

- $e^+e^- \to 3j$ ✔
- $ep \to (2+1)j$ ✘
- $pp \to j + X$ ✘
- $pp \to V$ ✔
- $pp \to V + j$ ✘
- $pp \to t\bar{t}$ ✔/✘

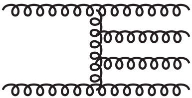
Processes with potentially large radiative corrections

- $pp \to H$ ✔
- $pp \to H + j$ ✘
- $pp \to VH$ ✔ $(V = W)$
- $pp \to VV$ ✔ $(VV = \gamma\gamma)$

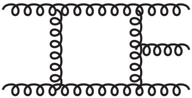
NNLO ingredients

A generic *m*-jet cross section at NNLO involves

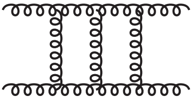
- Tree-level squared matrix elements
  - with $m+2$ parton kinematics
  - known from LO calculations
  - 'doubly-real' contribution (RR)



- One-loop squared matrix elements
  - with $m+1$ parton kinematics
  - usually known from NLO calculations
  - 'real-virtual' contribution (RV)



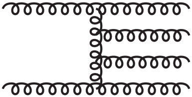
- Two-loop squared matrix elements
  - with m parton kinematics
  - known for all massless $2 \to 2$ processes
  - 'doubly-virtual' contribution (VV)



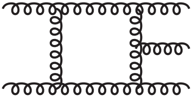
NNLO ingredients

A generic m-jet cross section at NNLO involves

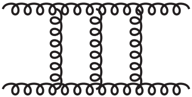
- Tree-level squared matrix elements
  - with $m+2$ parton kinematics
  - known from LO calculations
  - 'doubly-real' contribution (RR)



- One-loop squared matrix elements
  - with $m+1$ parton kinematics
  - usually known from NLO calculations
  - 'real-virtual' contribution (RV)



- Two-loop squared matrix elements
  - with m parton kinematics
  - known for all massless $2 \to 2$ processes
  - 'doubly-virtual' contribution (VV)



Assuming we know the relevant matrix elements, can we use those matrix elements to compute cross sections?

The problem - IR singularities

Consider the NNLO correction to a generic m-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m} \,.$$

Doubly-real

- $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2}$
- Tree MEs with $m+2$-parton kinematics
- kin. singularities as one or two partons unresolved: up to $\mathrm{O}(\epsilon^{-4})$ poles from PS integration
- no explicit ϵ poles

Real-virtual

- $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1}$
- One-loop MEs with $m+1$-parton kinematics
- kin. singularities as one parton unresolved: up to $\mathrm{O}(\epsilon^{-2})$ poles from PS integration
- explicit ϵ poles up to $\mathrm{O}(\epsilon^{-2})$

Doubly-virtual

- $\mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}$
- One- and two-loop MEs with m-parton kinematics
- kin. singularities screened by jet function: PS integration finite
- explicit ϵ poles up to $\mathrm{O}(\epsilon^{-4})$

The problem - IR singularities

Consider the NNLO correction to a generic m-jet observable

$$\sigma^{\mathrm{NNLO}} = \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} + \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1} + \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}\,.$$

THE KLN THEOREM

Infrared singularities cancel between real and virtual quantum corrections at the same order in perturbation theory, for sufficiently inclusive (i.e. IR safe) observables.

HOWEVER

How to make this cancellation explicit, so that the various contributions can be computed numerically? Need a method to deal with implicit poles.

Approaches

Sector decomposition

(Binoth, Heinrich; Anastasiou, Melnikov, Petriello; Czakon)

- extract ϵ poles of each contribution (RR, RV, VV) separately by expanding the integrand in distributions
- resulting expansion coefficients are finite multi-dimensional integrals, integrate numerically
- cancellation of poles numerical, depends on observable
- first method to yield physical results, but can it handle complicated final states?

Subtraction

(Catani, Grazzini; Cieri, Ferrera, de Florian; Gehrmann, Gehrmann-De Ridder, Glover; Weinzierl; Del Duca, Trócsányi, GS)

- rearrange the poles between real and virtual contributions by subtracting and adding back suitable approximate cross sections
- cancellation of explicit ϵ poles achieved analytically, remaining PS integrals are finite
- nice properties (generality, efficiency) expected form experience at NLO
- definition of subtraction terms is not unique, hence several approaches: $q_\perp$, antenna, local

Approaches

Sector decomposition

(Binoth, Heinrich, Anastasiou, Dixon, Melnikov, Petriello, Czakon)

- ✔ first method to yield physical cross sections
- ✔ cancellation of divergences fully numerical
- ✘ cancellation of poles also, and depends on jet function
- ✘ can it handle complicated final states?

$q_\perp$ subtraction

(Catani, Grazzini, Cieri, Ferrera, de Florian, Tramontano)

- ✔ exploits universal behavior of $q_\perp$ distribution at small $q_\perp$
- ✔ efficient and fully exclusive calculation
- ✘ limited scope: applicable only to production of massive colorless final states in hadron collisions

Antenna subtraction

(Gehrmann, Gehrmann-De Ridder, Glover, Heinrich, Weinzierl)

- ✔ successfully applied to $e^+e^- \to 2, 3j$
- ✔ analytic integration of antennae over unresolved phase space is understood
- ✘ counterterms are nonlocal
- ✘ treatment of color is implicit
- ✘ cannot cut factorized phase space

Approaches - two new developments

Refinement of the sector decomposition algorithm

Anastasiou, Lazopoulos, Herzog (2010)

- uses non-linear mappings to disentangle overlapping singularities
- the aim is to increase efficiency by reducing the large number of sectors/terms generated during decomposition
- first application: fully exclusive $H \to b\bar{b}$ decay at NNLO

Anastasiou, Lazopoulos, Herzog (2011)

Refinement of phase space integration via sector decomposition

Czakon (2010); Boughezal, Melnikov, Petriello (2011)

- FKS-like approach to double real radiation in $t\bar{t}$ production
- sector decomposition used to make singular contributions explicit, guided by known universal IR structure
- first NNLO computation of $q\bar{q} \to t\bar{t}$ total cross section

Baernreuther, Czakon, Mitov (2012)

Why a new scheme?

Goal: devise a subtraction scheme with

- general and explicit expressions, including color
  (view towards automation, color space notation is used)
- fully local counterterms, taking account of all color and spin correlations
  (mathematical rigor, efficiency)
- option to constrain subtractions to near singular regions
  (efficiency, important check)
- very algorithmic construction
  (valid at any order in perturbation theory)

Basics of subtraction

Subtraction - a caricature

Want to evaluate (at $\epsilon \to 0$)

$$\sigma = \int_0^1 \mathrm{d}\sigma^{\mathrm{R}}(x) + \sigma^{\mathrm{V}} \qquad \text{where} \qquad \begin{aligned} \mathrm{d}\sigma^{\mathrm{R}}(x) &= x^{-1-\epsilon} R(x) \\ R(0) &= R_0 < \infty \\ \sigma^{\mathrm{V}} &= R_0/\epsilon + V \end{aligned}$$

- define the counterterm

$$\mathrm{d}\sigma^{\mathrm{R,A}}(x) = x^{-1-\epsilon} R_0$$

- use it to reshuffle singularities between R and V contributions

$$\begin{aligned} \sigma &= \int_0^1 \left[\mathrm{d}\sigma^{\mathrm{R}}(x) - \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} + \left[\sigma^{\mathrm{V}} + \int_0^1 \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} \\ &= \int_0^1 \left[\frac{R(x) - R_0}{x^{1+\epsilon}}\right]_{\epsilon=0} + \left[\frac{R_0}{\epsilon} + V - \frac{R_0}{\epsilon}\right]_{\epsilon=0} \\ &= \int_0^1 \frac{R(x) - R_0}{x} + V \end{aligned}$$

The last integral is finite, computable with standard numerical methods.

The issue of locality

In a rigorous mathematical sense, the cancellation of both kinematical singularities and ϵ-poles must be local. I.e. the subtraction term must have the following general properties

- it must match the singularity structure of (singly- and doubly-) real emissions pointwise, in *d* dimensions
- its integrated form must be combined with the (real- and doubly-) virtual cross section explicitly, before phase space integration; ϵ-poles must cancel point by point

What about singular terms in the real emission cross section that cancel upon phase space integration (e.g. azimuthal correlations in gluon splitting)?

- they cancel upon integration in *d* dimensions, the corresponding four dimensional integrals are ill-defined
- it is mandatory to treat these terms, since naive numerical integration (in four dimensions) can give any result whatsoever
- however, can be treated with methods other than strict local subtraction, e.g. auxiliary phase space slicing (as in antenna subtraction)

A more efficient subtraction scheme?

Want to evaluate (at $\epsilon \to 0$)

$$\sigma = \int_0^1 \mathrm{d}\sigma^{\mathrm{R}}(x) + \sigma^{\mathrm{V}} \qquad \text{where} \qquad \begin{aligned} \mathrm{d}\sigma^{\mathrm{R}}(x) &= x^{-1-\epsilon} R(x) \\ R(0) &= R_0 < \infty \\ \sigma^{\mathrm{V}} &= R_0/\epsilon + V \end{aligned}$$

- define the counterterm

$$\mathrm{d}\sigma^{\mathrm{R,A}}(x) = x^{-1-\epsilon} R_0$$

- use it to reshuffle singularities between R and V contributions

$$\begin{aligned} \sigma &= \int_0^1 \left[\mathrm{d}\sigma^{\mathrm{R}}(x) - \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} + \left[\sigma^{\mathrm{V}} + \int_0^1 \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} \\ &= \int_0^1 \left[\frac{R(x) - R_0}{x^{1+\epsilon}}\right]_{\epsilon=0} + \left[\frac{R_0}{\epsilon} + V - \frac{R_0}{\epsilon}\right]_{\epsilon=0} \\ &= \int_0^1 \frac{R(x) - R_0}{x} + V \end{aligned}$$

The last integral is finite, computable with standard numerical methods.

A more efficient subtraction scheme?

Want to evaluate (at $\epsilon \to 0$)

$$\sigma = \int_0^1 \mathrm{d}\sigma^{\mathrm{R}}(x) + \sigma^{\mathrm{V}} \qquad \text{where} \qquad \begin{aligned} \mathrm{d}\sigma^{\mathrm{R}}(x) &= x^{-1-\epsilon} R(x) \\ R(0) &= R_0 < \infty \\ \sigma^{\mathrm{V}} &= R_0/\epsilon + V \end{aligned}$$

- define the counterterm to be nonzero only near the singular region

$$\mathrm{d}\sigma^{\mathrm{R,A}}(x) = x^{-1-\epsilon} R_0\, \Theta(x_0 - x)$$

- use it to reshuffle singularities between R and V contributions

$$\begin{aligned} \sigma &= \int_0^1 \left[\mathrm{d}\sigma^{\mathrm{R}}(x) - \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} + \left[\sigma^{\mathrm{V}} + \int_0^1 \mathrm{d}\sigma^{\mathrm{R,A}}(x)\right]_{\epsilon=0} \\ &= \int_0^1 \left[\frac{R(x) - R_0\Theta(x_0 - x)}{x^{1+\epsilon}}\right]_{\epsilon=0} + \left[\frac{R_0}{\epsilon} + V - \frac{R_0}{\epsilon} + R_0 \log x_0 + \mathrm{O}(\epsilon^1)\right]_{\epsilon=0} \\ &= \int_0^1 \frac{R(x) - R_0\Theta(x_0 - x)}{x} + V + R_0 \log x_0 \end{aligned}$$

The last integral is finite, computable with standard numerical methods.

A more efficient subtraction scheme?

It is sufficient to perform subtraction only near the singular region

- ✔ gain in efficiency: subtraction term only needs to be computed over a fraction of phase space
- ✔ strong check: final result is independent of value of phase space cut
- ✘ analytical integration of subtraction term more difficult (extra scale involved)

Local subtraction at NNLO

Structure of the NNLO correction

Rewrite the NNLO correction as a sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

Structure of the NNLO correction

Rewrite the NNLO correction as a sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$

Structure of the NNLO correction

Rewrite the NNLO correction as a sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$

Structure of the NNLO correction

Rewrite the NNLO correction as a sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
3. $\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}$ accounts for the overlap of $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$

Structure of the NNLO correction

Rewrite the NNLO correction as a sum of three terms

$$\sigma^{\rm NNLO} = \sigma^{\rm RR}_{m+2} + \sigma^{\rm RV}_{m+1} + \sigma^{\rm VV}_{m} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m}$$

each integrable in four dimensions

$$\sigma^{\rm NNLO}_{m+2} = \int_{m+2} \left\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \left[{\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\rm NNLO}_{m+1} = \int_{m+1} \left\{ \left[{\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right] J_{m+1} - \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] J_m \right\}$$

$$\sigma^{\rm NNLO}_{m} = \int_{m} \left\{ {\rm d}\sigma^{\rm VV}_{m} + \int_2 \left[{\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \right] + \int_1 \left[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \left(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \right)^{\rm A_1} \right] \right\} J_m$$

1. ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ regularizes the doubly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$
2. ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$ regularizes the singly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$
3. ${\rm d}\sigma^{\rm RR,A_{12}}_{m+2}$ accounts for the overlap of ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$ and ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$
4. ${\rm d}\sigma^{\rm RV,A_1}_{m+1}$ regularizes the singly-unresolved limits of ${\rm d}\sigma^{\rm RV}_{m+1}$

Structure of the NNLO correction

Rewrite the NNLO correction as a sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \boxed{\left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \boxed{\left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}}} \right] \right\} J_m$$

1. $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ regularizes the doubly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
2. $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RR}}_{m+2}$
3. $\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}$ accounts for the overlap of $\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$ and $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$
4. $\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1}$ regularizes the singly-unresolved limits of $\mathrm{d}\sigma^{\mathrm{RV}}_{m+1}$
5. $\boxed{\left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}\right)^{\mathrm{A_1}}}$ regularizes the singly-unresolved limit of $\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}$

Defining a subtraction scheme

Strategy: IR limits are process independent and known

1. Start from defining the subtraction terms based on IR limit formulae
   - they are trivially general, explicit and local
   - done some time ago (2006) for colorless initial states
2. Worry about integrating them later
   - since this is *in principle* a very narrowly defined problem, given 1.
   - but in practice is very cumbersome, due to lack of technology

Defining a subtraction scheme

The following three problems must be addressed

1. Matching of limits to avoid multiple subtraction in overlapping singular regions of PS. Easy at NLO: collinear limit + soft limit - collinear limit of soft limit.

$$\mathbf{A}_1|\mathcal{M}_{m+1}^{(0)}|^2 = \sum_i \left[\sum_{i\neq r}\frac{1}{2}\mathbf{C}_{ir} + \mathbf{S}_r - \sum_{i\neq r}\mathbf{C}_{ir}\mathbf{S}_r\right]|\mathcal{M}_{m+1}^{(0)}|^2$$

2. Extension of IR factorization formulae over full PS using momentum mappings that respect factorization and delicate structure of cancellations in all limits.

$$\{p\}_{m+1} \xrightarrow{r} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+1}(\{p\}_{m+1}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m; Q)[\mathrm{d}p_{1,m}]$$
$$\{p\}_{m+2} \xrightarrow{r,s} \{\tilde{p}\}_m : \quad \mathrm{d}\phi_{m+2}(\{p\}_{m+2}; Q) = \mathrm{d}\phi_m(\{\tilde{p}\}_m; Q)[\mathrm{d}p_{2,m}]$$

3. Integration of the counterterms over the phase space of the unresolved parton(s).

The need for extension

IR limit formulae are only well-defined in the strict limit. E.g.

- collinear: $\mathbf{C}_{ir}$ is a symbolic operator that takes the $p_i || p_r$ limit

$$\mathbf{C}_{ir} |\mathcal{M}^{(0)}_{m+2}(p_i, p_r, \ldots)|^2 = 8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon} \frac{1}{s_{ir}} \hat{P}_{f_i f_r}(z_i, z_r, k_\perp; \epsilon) \otimes |\mathcal{M}^{(0)}_{m+1}(p_{ir}, \ldots)|^2$$

- soft: $\mathbf{S}_r$ is a symbolic operator that takes the $p_r \to 0$ limit

$$\mathbf{S}_r |\mathcal{M}^{(0)}_{m+2}(p_r, \ldots)|^2 = -8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon} \sum_{i,k} \frac{1}{2} \mathcal{S}_{ik}(r) |\mathcal{M}^{(0)}_{m+1,(i,k)}(\cancel{p_r}, \ldots)|^2$$

NOTICE

- momenta in factorized ME's on the r.h.s. conserve momentum and/or mass shell conditions only in the strict limit
- arguments of AP splitting functions, e.g. momentum fractions z_i, z_r and transverse momentum $k_\perp$ are only defined in the strict limit

HENCE

- must specify precisely momenta entering factorized ME's away from limit
- must define z_i, z_r and $k_\perp$ away from limit

Defining a subtraction scheme

Specific issues at NNLO

1. Matching is cumbersome if done in a brute force way. However, an efficient solution that works at any order in PT is known.
2. Extension is delicate. E.g. counterterms for singly-unresolved real emission (unintegrated and integrated) must have universal IR limits. This is not guaranteed by QCD factorization.
3. Choosing the counterterms such that integration is (relatively) straightforward generally conflicts with the delicate cancellation of IR singularities.

NNLO subtraction terms - an example

MESSAGE

- Subtraction terms are defined completely explicitly for any number of jets.

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A_2}}$ we find

$$\mathcal{C}_{ir;js}^{(0,0)}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$

$$\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_if_r}(z_{r,i},z_{i,r},k_{\perp,i,r};\epsilon)\hat{P}_{f_jf_s}(z_{s,j},z_{j,s},k_{\perp,j,s};\epsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})\rangle$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2}$ we find

$$\mathcal{C}_{ir;js}^{(0,0)}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{\boxed{s_{ir}s_{js}}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$

$$\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_i f_r}(z_{r,i}, z_{i,r}, k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j}, z_{j,s}, k_{\perp,j,s};\epsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})\rangle$$

➠ collinear poles: $\boxed{s_{ir}s_{js}}$

$$s_{kl} = 2p_k \cdot p_l\,, \qquad k,l = i,r \quad \text{or} \quad j,s$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2}$ we find

$$\mathcal{C}_{ir;js}^{(0,0)}(\{p\}) = (8\pi\alpha_s\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$
$$\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})|\boxed{\hat{P}_{f_i f_r}(z_{r,i},z_{i,r},k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j},z_{j,s},k_{\perp,j,s};\epsilon)}|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})\rangle$$

- Altarelli-Parisi splitting functions: $\boxed{\hat{P}_{f_i f_r}\hat{P}_{f_j f_s}}$

$$z_{k,l} = \frac{y_{kQ}}{y_{(kl)Q}}\,, \qquad k_{\perp,k,l}^{\mu} = \zeta_{k,l}p_l^{\mu} - \zeta_{l,k}p_k^{\mu} + \zeta_{kl}\tilde{p}_{kl}^{\mu}\,, \qquad k,l = i,r \quad \text{or} \quad j,s$$

with

$$\zeta_{k,l} = z_{k,l} - \frac{y_{kl}}{\alpha_{kl}y_{(kl)Q}}\,, \qquad \zeta_{kl} = \frac{y_{kl}}{\alpha_{kl}y_{\widetilde{kl}Q}}(z_{l,k}-z_{k,l})$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2}$ we find

$$\mathcal{C}^{(0,0)}_{ir;js}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$
$$\times \langle \mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_i f_r}(z_{r,i},z_{i,r},k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j},z_{j,s},k_{\perp,j,s};\epsilon)|\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir;js)})\rangle$$

- mapped momenta: $\{\tilde{p}\}^{(ir;js)} = \{\tilde{p}_1,\ldots,\tilde{p}_{ir},\ldots,\tilde{p}_{js},\ldots,\tilde{p}_{m+2}\}_m$

$$\tilde{p}^\mu_{kl} = \frac{1}{1-\alpha_{ir}-\alpha_{js}}(p^\mu_k + p^\mu_l - \alpha_{kl}Q^\mu)\,, \qquad k,l = i,r \quad \text{or} \quad j,s$$

$$\tilde{p}^\mu_n = \frac{1}{1-\alpha_{ir}-\alpha_{js}}p^\mu_n\,, \qquad n \neq i,r,j,s$$

with

$$\alpha_{kl} = \frac{1}{2}\left[y_{(kl)Q} - \sqrt{y^2_{(kl)Q} - 4y_{kl}}\,\right]\,, \qquad k,l = i,r \quad \text{or} \quad j,s$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$ we find

$$\mathcal{C}_{ir;js}^{(0,0)}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\boxed{\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})}$$
$$\times \langle\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_i f_r}(z_{r,i}, z_{i,r}, k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j}, z_{j,s}, k_{\perp,j,s};\epsilon)|\mathcal{M}_m^{(0)}(\{\tilde{p}\}^{(ir;js)})\rangle$$

➠ constrain subtraction to near singular region: $\boxed{\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})}$

$0 < \alpha_0 \le 1\,,\qquad \alpha_0 = 1$: subtract over full phase space

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2}$ we find

$$\mathcal{C}^{(0,0)}_{ir;js}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}} \boxed{(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}} \Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$
$$\times \langle \mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_i f_r}(z_{r,i}, z_{i,r}, k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j}, z_{j,s}, k_{\perp,j,s};\epsilon)|\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir;js)})\rangle$$

➠ make integrated counterterm m-independent: $\boxed{(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}}$

$$d(m;\epsilon) = 2m(1-\epsilon) - 2d_0\,, \qquad d_0 = D_0 + d_1\epsilon\,, \qquad D_0 \geq 2$$

NNLO subtraction terms - an example

Double collinear counterterm: among others, in $\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2}$ we find

$$\mathcal{C}^{(0,0)}_{ir;js}(\{p\}) = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \frac{1}{s_{ir}s_{js}}(1-\alpha_{ir}-\alpha_{js})^{-d(m;\epsilon)}\Theta(\alpha_0-\alpha_{ir}-\alpha_{js})$$
$$\times \langle \mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir;js)})|\hat{P}_{f_i f_r}(z_{r,i}, z_{i,r}, k_{\perp,i,r};\epsilon)\hat{P}_{f_j f_s}(z_{s,j}, z_{j,s}, k_{\perp,j,s};\epsilon)|\mathcal{M}^{(0)}_m(\{\tilde{p}\}^{(ir;js)})\rangle$$

The complete approximate cross section is a sum of such terms

$$\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} = \mathrm{d}\phi_m[\mathrm{d}p_2]\boldsymbol{\mathcal{A}}_2|\mathcal{M}^{(0)}_{m+2}|^2$$

where

$$\boldsymbol{\mathcal{A}}_2|\mathcal{M}^{(0)}_{m+2}|^2 = \sum_r\sum_{s\neq r}\Bigg\{\sum_{i\neq r,s}\left[\frac{1}{6}\mathcal{C}^{(0,0)}_{irs} + \sum_{j\neq i,r,s}\frac{1}{8}\mathcal{C}^{(0,0)}_{ir;js} + \frac{1}{2}\mathcal{CS}^{(0,0)}_{ir;s}\right] + \frac{1}{2}\mathcal{S}^{(0,0)}_{rs}$$
$$-\sum_{i\neq r,s}\left[\frac{1}{2}\mathcal{C}_{irs}\mathcal{CS}^{(0,0)}_{ir;s} + \sum_{j\neq i,r,s}\frac{1}{2}\mathcal{C}_{ir;js}\mathcal{CS}^{(0,0)}_{ir;s} + \frac{1}{2}\mathcal{C}_{irs}\mathcal{S}^{(0,0)}_{rs} + \mathcal{CS}_{ir;s}\mathcal{S}^{(0,0)}_{rs}\right.$$
$$\left.\left.-\sum_{j\neq i,r,s}\frac{1}{2}\mathcal{C}_{ir;js}\mathcal{S}^{(0,0)}_{rs} - \mathcal{C}_{irs}\mathcal{CS}_{ir;s}\mathcal{S}^{(0,0)}_{rs}\right]\right\}$$

NNLO subtraction terms - general features

Based on universal IR limit formulae

- Altarelli-Parisi splitting functions, soft currents (tree and one-loop, triple AP functions)
- simple and general procedure for matching of limits using physical gauge
- extension based on momentum mappings that can be generalized to any number of unresolved partons

Fully local in color $\otimes$ spin space

- no need to consider the color decomposition of real emission ME's
- azimuthal correlations correctly taken into account in gluon splitting
- can check explicitly that the ratio of the sum of counterterms to the real emission cross section tends to unity in any IR limit

Straightforward to constrain subtractions to near singular regions

- gain in efficiency
- independence of physical results on phase space cut is strong check

Given completely explicitly for any process with non colored initial state

The tedious part: integrating the counterterms

Basic setup

Momentum mappings used to define the counterterms

$$\{p\}_{n+p} \xrightarrow{R} \{\tilde{p}\}_n$$

- implement exact momentum conservation
- recoil distributed democratically (can be generalized to any p)
- different collinear and soft mappings (R labels precise limit)
- exact factorization of phase space

$$\mathrm{d}\phi_{n+p}(\{p\};Q) = \mathrm{d}\phi_n(\{\tilde{p}\}_n^{(R)};Q)[\mathrm{d}p_{p,n}^{(R)}]$$

Counterterms are products (in color and spin space) of

- factorized ME's independent of variables in $[\mathrm{d}p_{p,n}^{(R)}]$
- singular factors (AP functions, soft currents), to be integrated over $[\mathrm{d}p_{p,n}^{(R)}]$

Strategy for computing the integrals

- explicit parametrization of factorized phase space leads to parametric integral representations
- evaluate the parametric integrals

Types of integrated counterterms

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

Types of integrated counterterms

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \boxed{\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2}} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\boxed{\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1}} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

➠ tree-level and one-loop singly-unresolved integrals

Types of integrated counterterms

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \boxed{\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \boxed{\left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}}} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals

Types of integrated counterterms

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\boxed{\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2}} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} \right)^{\mathrm{A}_1} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

Types of integrated counterterms

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \boxed{\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

Phase space integrals - an example

MESSAGE

- The integral is (very) difficult, but the result is numerically (very) simple.

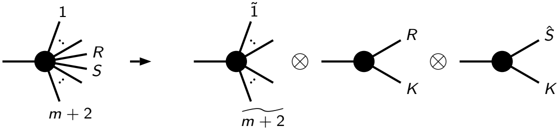
Phase space integrals - an example

Abelian soft-double soft counterterm: among many others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$ we find

$$\left(\mathcal{S}_t\mathcal{S}_{rt}^{(0)}\right)^{\mathrm{ab}} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \sum_{i,j,k,l} \frac{1}{8}\mathcal{S}_{\hat{i}\hat{k}}(\hat{r})\mathcal{S}_{jl}(t)|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$
$$\times (1-y_{tQ})^{d_0'-m(1-\epsilon)}(1-y_{\hat{r}Q})^{d_0'-m(1-\epsilon)}\Theta(y_0-y_{tQ})\Theta(y_0-y_{\hat{r}Q})$$

The set of m momenta, $\{\tilde{p}\}$, is obtained by an iterated mapping which leads to an exact factorization of phase space

$$\{p\}_{m+2} \xrightarrow{\mathsf{S}_t} \{\hat{p}\}_{m+1} \xrightarrow{\mathsf{S}_{\hat{r}}} \{\tilde{p}\} : \quad \mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_m(\{\tilde{p}\};Q)[\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]$$



Phase space integrals - an example

Abelian soft-double soft counterterm: among many others, in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_{12}}$ we find

$$\left(\mathcal{S}_t\mathcal{S}_{rt}^{(0)}\right)^{\mathrm{ab}} = (8\pi\alpha_{\mathrm{s}}\mu^{2\epsilon})^2 \sum_{i,j,k,l} \frac{1}{8}\mathcal{S}_{\hat{i}\hat{k}}(\hat{r})\mathcal{S}_{jl}(t)|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$
$$\times (1-y_{tQ})^{d_0'-m(1-\epsilon)}(1-y_{\hat{r}Q})^{d_0'-m(1-\epsilon)}\Theta(y_0-y_{tQ})\Theta(y_0-y_{\hat{r}Q})$$

The set of m momenta, $\{\tilde{p}\}$, is obtained by an iterated mapping which leads to an exact factorization of phase space

$$\{p\}_{m+2} \xrightarrow{\mathsf{S}_t} \{\hat{p}\}_{m+1} \xrightarrow{\mathsf{S}_{\hat{r}}} \{\tilde{p}\} : \quad \mathrm{d}\phi_{m+2}(\{p\};Q) = \mathrm{d}\phi_m(\{\tilde{p}\};Q)[\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]$$

Then we must compute

$$\int [\mathrm{d}\widehat{p}_{1,m}][\mathrm{d}p_{1,m+1}]\mathcal{S}_t\mathcal{S}_{rt}^{(0)} \equiv \left[\frac{\alpha_{\mathrm{s}}}{2\pi}S_\epsilon\left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i,k,j,l}[\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl}|\mathcal{M}_{m,(i,k)(j,l)}^{(0)}(\{\tilde{p}\})|^2$$

where $[\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl} \equiv [\mathrm{S}_t\mathrm{S}_{rt}^{(0)}]_{ikjl}(p_i,p_k,p_j,p_l,\epsilon,y_0,d_0')$ is a kinematics dependent function.

Abelian soft-double soft integral

For simplicity, consider the terms in the sum where $j = i$ and $l = k$: $[\mathrm{S}_t \mathrm{S}_{rt}^{(0)}]_{ikik}$. Kinematical dependence is through $\cos\chi_{ik} = \measuredangle(p_i, p_k)$, we set $\cos\chi_{ik} = 1 - 2Y_{ik,Q}$.

Using angles and energies in some specific Lorentz frame to parametrize the factorized phase space measures, $[\mathrm{d}\widehat{p}_{1,m}]$ and $[\mathrm{d}p_{1,m+1}]$, we find that $[\mathrm{S}_t \mathrm{S}_{rt}^{(0)}]_{ikik}$ is proportional to

$$\mathcal{I}_{\mathcal{S}}^{(11)}(Y_{ik,Q}; \epsilon, y_0, d_0') = -\frac{4\Gamma^4(1-\epsilon)}{\pi\Gamma^2(1-\epsilon)} \frac{B_{y_0}(-2\epsilon, d_0'+1)}{\epsilon} Y_{ik,Q} \int_0^{y_0} \mathrm{d}y\, y^{-1-2\epsilon}(1-y)^{d_0'-1+\epsilon}$$
$$\times \int_{-1}^{1} \mathrm{d}(\cos\vartheta)\,(\sin\vartheta)^{-2\epsilon} \int_{-1}^{1} \mathrm{d}(\cos\varphi)\,(\sin\varphi)^{-1-2\epsilon} \big[f(\vartheta,\varphi;0)\big]^{-1} \big[f(\vartheta,\varphi;Y_{ik,Q})\big]^{-1}$$
$$\times \big[Y(y,\vartheta,\varphi;Y_{ik,Q})\big]^{-\epsilon} {}_2F_1\big(-\epsilon, -\epsilon, 1-\epsilon, 1 - Y(y,\vartheta,\varphi;Y_{ik,Q})\big)$$

where

$$f(\vartheta,\varphi;Y_{ik,Q}) = 1 - 2\sqrt{Y_{ik,Q}(1-Y_{ik,Q})}\sin\vartheta\cos\varphi - (1-2Y_{ik,Q})\chi\cos\vartheta$$

$$Y(y,\vartheta,\varphi;\chi) = \frac{4(1-y)Y_{ik,Q}}{[2(1-y) + y\, f(\vartheta,\varphi;0)][2(1-y) + y\, f(\vartheta,\varphi;Y_{ik,Q})]}$$

Abelian soft-double soft integral

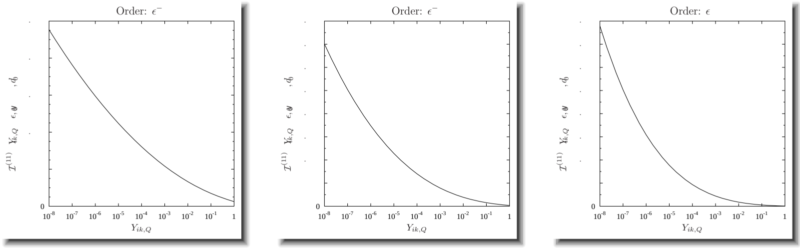
This integral is equal to

$$\mathcal{I}_{\mathcal{S}}^{(11)}(Y_{ik,Q};\epsilon,y_0,d'_0) = \frac{1}{\epsilon^4} - 2\Big[\ln(Y_{ik,Q}) + \Sigma(y_0,D'_0) + \Sigma(y_0,D'_0-1)\Big]\frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2})$$

where $D'_0 = d'_0|_{\epsilon=0}$ and the dependence on the PS cut parameter, y_0, enters in

$$\Sigma(z,N) = \ln z - \sum_{k=1}^{N} \frac{1-(1-z)^k}{k}$$

Higher order expansion coefficients computed numerically $(y_0 = 1,\ D'_0 = 3)$



Analytical vs. numerical

As a matter of principle

- Rigorous proof of cancellation of IR poles requires poles of integrated counterterms in analytical form.
- Analytical forms are fast and accurate compared to numerical ones.

However

- Analytical results show (in all cases where they are available) that integrated counterterms are smooth functions of kinematic variables.

Hence

- Numerical forms of integrated counterterms are sufficient for practical purposes. Final results can be conveniently given by interpolating tables or approximating functions computed once and for all. Thus, efficient implementation is possible even if the full analytical calculation is not feasible or practical (e.g. finite parts of integrated counterterms).
- In particular, suitable approximating functions may be obtained by fitting.

Example of approximation by fitting

Doubly-unresolved double-collinear master integral $\mathcal{I}_{2\mathcal{C}}(x_{\widetilde{ir}}, x_{\widetilde{js}}; \epsilon, , 3-3\epsilon, k, l)$

$$
\begin{aligned}
\mathcal{I}_{2\mathcal{C}}(x_{\widetilde{ir}}, x_{\widetilde{js}}; \epsilon, \alpha_0, d_0; k, l) &= x_{\widetilde{ir}} x_{\widetilde{js}} \int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta \int_0^1 \mathrm{d}v\,\mathrm{d}u\,\alpha^{-1-\epsilon}\beta^{-1-\epsilon}(1-\alpha-\beta)^{2d_0-2(1-\epsilon)} \\
&\times [\alpha+(1-\alpha-\beta)x_{\widetilde{ir}}]^{-1-\epsilon}[\beta+(1-\alpha-\beta)x_{\widetilde{js}}]^{-1-\epsilon} v^{-\epsilon}(1-v)^{-\epsilon}u^{-\epsilon}(1-u)^{-\epsilon} \\
&\times \left(\frac{\alpha+(1-\alpha-\beta)x_{\widetilde{ir}}v}{2\alpha+(1-\alpha-\beta)x_{\widetilde{ir}}}\right)^k \left(\frac{\beta+(1-\alpha-\beta)x_{\widetilde{js}}u}{2\beta+(1-\alpha-\beta)x_{\widetilde{js}}}\right)^l \Theta(\alpha_0-\alpha-\beta)
\end{aligned}
$$

Example of approximation by fitting

Doubly-unresolved double-collinear master integral $\mathcal{I}_{2\mathcal{C}}(x_{\widetilde{ir}}, x_{\widetilde{js}}; \epsilon, , 3-3\epsilon, k, l)$

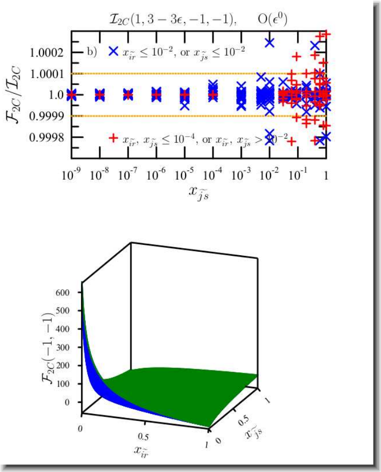
- poles (up to $\mathrm{O}(\epsilon^{-4})$) extracted via sector decomposition
- numerical values of pole coefficients computed for a 17×17 grid with precision of $\sim 10^{-7}$
- define three regions (note: result is symmetric in $x_{\widetilde{ir}}, x_{\widetilde{js}}$)
  - asymptotic: $x_{\widetilde{ir}}, x_{\widetilde{js}} < 10^{-4}$
  - non-asymptotic: $x_{\widetilde{ir}}, x_{\widetilde{js}} > 10^{-2}$
  - border: $x_{\widetilde{ir}} < 10^{-2}$ or $x_{\widetilde{js}} < 10^{-2}$
- in each region, fit with ansatz

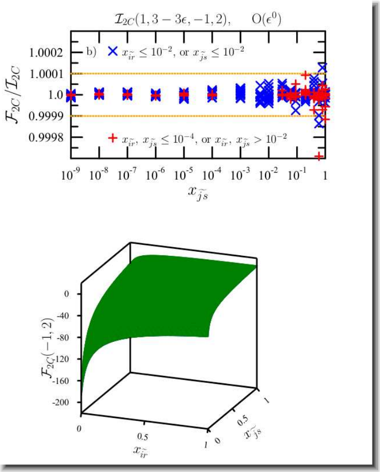
$$\mathcal{F}(x_1, x_2) = \sum_{p_i, l_i} C_{m;p_1,p_2;l_1,l_2}(x_1^{p_1} x_2^{p_2})(\log^{l_1}(x_1) \log^{l_2}(x_2))$$

where $p_1 + p_2 \leq m$ with m a free parameter, while $l_1 + l_2 \leq n$ and n is predicted

Example of approximation by fitting

Doubly-unresolved double-collinear master integral $\mathcal{I}_{2C}(x_{\widetilde{ir}}, x_{\widetilde{js}}; \epsilon, , 3-3\epsilon, k, l)$





Phase space integrals - methods

Several different methods to compute the integrals have been explored

- use of IBPs to reduce to master integrals + solution of MIs by differential equations
- use of MB representations to extract pole structure + summation of nested series
- use of sector decomposition

Phase space integrals - methods

Method	Analytical	Numerical
IBP	✔ singly-unresolved integrals ✘ bottleneck is the proliferation of denominators	✔ by evaluating the analytic expressions ✘ no numbers without full analytical results
MB	✔ iterated singly-unresolved integrals ✘ bottleneck is the evaluation of sums	✔ direct numerical evaluation of MB integrals possible ✔ fast and accurate
SD	✔ easy to automate ✘ only in principle, except for lowest order poles	✘ numerical behavior is generally worse than MB method (speed, accuracy)

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

(GS, J. Math. Phys. **52** (2011) 083501.)

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables.

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

(GS, J. Math. Phys. **52** (2011) 083501.)

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_s}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\mathbf{v} = (v_{11}, v_{12}, \ldots, v_{1n}, v_{22}, v_{23}, \ldots, v_{n-1n}, v_{nn}), \qquad v_{kl} \equiv \begin{cases} \dfrac{p_k \cdot p_l}{2} & ; \quad k \neq l \\ \dfrac{p_k^2}{4} & ; \quad k = l \end{cases}$$

$$\boldsymbol{\alpha} = (\mathbf{0}_N, j_1, \ldots, j_n, 1 - j - \epsilon), \qquad \boldsymbol{\beta} = (j_1, \ldots, j_n, 2 - j - 2\epsilon)$$

and $\mathbf{L_s} = L_{s_1} \times \ldots L_{s_N}$, where L_{s_k} is an infinite contour in the complex s_k-plane running from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$.

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

(GS, J. Math. Phys. **52** (2011) 083501.)

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1 \cdot q)^{j_1} \cdots (p_n \cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon} \pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_s}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\mathbf{A} = \left[\begin{array}{c} -\mathbf{1}_{N\times N} \\ \hline \mathbf{M}_{n\times N} \\ \hline -1 \cdots -1 \end{array}\right], \qquad \mathbf{B} = [(0)_{(n+1)\times N}]$$

i.e. **B** is zero, while the $n \times N$ dimensional matrix **M** has the block form:

$$\mathbf{M}_{n\times N} = \left[\ \mathbf{m}_{n\times n}\ \middle|\ \mathbf{m}_{n\times(n-1)}\ \middle|\ \cdots\ \middle|\ \mathbf{m}_{n\times 1}\ \right] \quad \text{with} \quad \mathbf{m}_{n\times p} = \left[\begin{array}{c|c} 0 & (0)_{(n-p)\times(p-1)} \\ \hline 2 & 1\cdots 1 \\ \hline 0 & \\ \vdots & \mathbf{1}_{(p-1)\times(p-1)} \\ 0 & \end{array}\right]$$

Spinoff - angular integrals in d dimensions

Consider the d dimensional angular integral with n denominators

(GS, J. Math. Phys. **52** (2011) 083501.)

$$\Omega_{j_1,\ldots,j_n} = \int \mathrm{d}\Omega_{d-1}(q) \frac{1}{(p_1\cdot q)^{j_1}\cdots(p_n\cdot q)^{j_n}}$$

We find (with $j = j_1 + \ldots + j_n$)

$$\Omega_{j_1,\ldots,j_n} = 2^{2-j-2\epsilon}\pi^{1-\epsilon} H[\mathbf{v}; (\boldsymbol{\alpha}, \mathbf{A}); (\boldsymbol{\beta}, \mathbf{B}); \mathbf{L_S}]$$

where H is the so-called H-function of $N = \frac{n(n+1)}{2}$ variables. We have

$$\Omega_{j_1,\ldots,j_n}(\{v_{kl}\};\epsilon) = 2^{2-j-2\epsilon}\pi^{1-\epsilon} \frac{1}{\prod_{k=1}^{n}\Gamma(j_k)\Gamma(2-j-2\epsilon)} \\ \times \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \left[\prod_{k=1}^{n}\prod_{l=k}^{n} \frac{\mathrm{d}z_{kl}}{2\pi\mathrm{i}} \Gamma(-z_{kl})\,(v_{kl})^{z_{kl}}\right] \left[\prod_{k=1}^{n}\Gamma(j_k+z_k)\right] \Gamma(1-j-\epsilon-z)\,.$$

where

$$z = \sum_{k=1}^{n}\sum_{l=k}^{n} z_{kl}\,, \qquad \text{and} \qquad z_k = \sum_{l=1}^{k} z_{lk} + \sum_{l=k}^{n} z_{kl}\,.$$

Integrated approximate cross sections

Structure of the results

Integrated approximate cross sections

- After summing over unobserved flavors, all integrated approximate cross sections can be written as products (in color space) of various insertion operators with lower point cross sections.

Insertion operators

- color and flavor structure of all insertion operators known
- first two leading poles of kinematical functions entering insertion operators known analytically in all cases (except $\mathbf{I}_2^{(0)}$)
- higher order expansion coefficients computed numerically

Integrated approximate cross sections - an example

MESSAGE

➠ Done once and for all (though admittedly lots of tedious work).

Integrated approximate cross sections - an example

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

Each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \left[\mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \boxed{\mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2}} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \right\} J_m$$

- tree-level and one-loop singly-unresolved integrals
- tree-level iterated singly-unresolved integrals
- tree-level doubly-unresolved integrals

Integrated approximate cross sections - an example

Iterated singly-unresolved

$$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} = \mathrm{d}\sigma_m^{\mathrm{B}} \otimes \boldsymbol{I}_{12}^{(0)}(\{p\}_m;\epsilon)$$

- structure of insertion operator in color $\otimes$ flavor space

$$\begin{aligned}\boldsymbol{I}_{12}^{(0)}(\{p\}_m;\epsilon) = \left[\frac{\alpha_{\mathrm{s}}}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ & \sum_i \left[\mathrm{C}_{12,f_i}^{(0)} \boldsymbol{T}_i^2 + \sum_k \mathrm{C}_{12,f_i f_k}^{(0)} \boldsymbol{T}_k^2\right] \boldsymbol{T}_i^2 \\ & + \sum_{j,l} \left[\mathrm{S}_{12}^{(0),(j,l)} C_{\mathrm{A}} + \sum_i \mathrm{CS}_{12,f_i}^{(0),(j,l)} \boldsymbol{T}_i^2\right] \boldsymbol{T}_j \boldsymbol{T}_l \\ & + \sum_{i,k,j,l} \mathrm{S}_{12}^{(0),(i,k)(j,l)} \{\boldsymbol{T}_i \boldsymbol{T}_k, \boldsymbol{T}_j \boldsymbol{T}_l\} \Bigg\}\end{aligned}$$

- $\mathrm{C}_{12,f_i}^{(0)}$, $\mathrm{C}_{12,f_i f_k}^{(0)}$, $\mathrm{S}_{12}^{(0),(j,l)}$, $\mathrm{CS}_{12,f_i}^{(0),(j,l)}$ and $\mathrm{S}_{12}^{(0),(i,k)(j,l)}$ are kinematical functions with poles up to $\mathrm{O}(\epsilon^{-4})$ (also depend on PS cut parameters)
- kinematical dependence through

$$x_i = y_{iQ} \equiv \frac{2p_i \cdot Q}{Q^2} \qquad \text{and} \qquad Y_{ik,Q} = \frac{y_{ik}}{y_{iQ} y_{kQ}}$$

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)

$$
\begin{aligned}
\mathbf{I}_{12}^{(0)}(p_1,p_2,p_3;\epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \Bigg\{ & \frac{6C_F^2 + 2C_A C_F + C_A^2}{\epsilon^4} + \Bigg[12C_F^2 + \frac{101 C_A C_F}{6} \\
& + \frac{67C_A^2}{12} - \frac{13C_F T_R n_f}{3} - \frac{3C_A T_R n_f}{2} - \left(8C_F^2 + C_A C_F - \frac{5C_A^2}{2}\right) \ln y_{12} \\
& - \left(4C_A C_F + \frac{5C_A^2}{2}\right)(\ln y_{13} + \ln y_{23}) - (4C_F^2 - 6C_A C_F - C_A^2)\Sigma(y_0, D_0') \\
& - (4C_F^2 - 4C_A C_F)\Sigma(y_0, D_0' - 1)\Bigg] \frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2}) \Bigg\}
\end{aligned}
$$

- notice x and Y dependence combine to produce just y_{ik} dependence, as expected
- dependence on PS cut parameters through

$$\Sigma(z, N) = \ln z - \sum_{k=1}^{N} \frac{1-(1-z)^k}{k}$$

should vanish once all integrated approximate cross sections are combined

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \left\{ \frac{6C_F^2 + 2C_A C_F + C_A^2}{\epsilon^4} + \left[12 C_F^2 + \frac{101 C_A C_F}{6} \right.\right.$$
$$+ \frac{67 C_A^2}{12} - \frac{13 C_F T_R n_f}{3} - \frac{3 C_A T_R n_f}{2} - \left(8C_F^2 + C_A C_F - \frac{5C_A^2}{2}\right) \boxed{\ln y_{12}}$$
$$- \left(4 C_A C_F + \frac{5C_A^2}{2}\right)\left(\boxed{\ln y_{13}} + \boxed{\ln y_{23}}\right) - (4C_F^2 - 6C_A C_F - C_A^2)\Sigma(y_0, D_0')$$
$$\left.\left. - (4C_F^2 - 4C_A C_F)\Sigma(y_0, D_0' - 1)\right] \frac{1}{\epsilon^3} + \mathrm{O}(\epsilon^{-2}) \right\}$$

- notice x and Y dependence combine to produce just $\boxed{y_{ik}}$ dependence, as expected
- dependence on PS cut parameters through

$$\Sigma(z, N) = \ln z - \sum_{k=1}^{N} \frac{1-(1-z)^k}{k}$$

should vanish once all integrated approximate cross sections are combined

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)
- higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1, p_2, p_3) + \text{O}(\epsilon^1)$$

- kinematical point parametrized by y_{ij}

$$y_{12} = 0.333333, \qquad y_{13} = 0.333333, \qquad y_{23} = 0.333333$$

Col	$\text{O}(\epsilon^{-4})$	$\text{O}(\epsilon^{-3})$	$\text{O}(\epsilon^{-2})$	$\text{O}(\epsilon^{-1})$	$\text{O}(\epsilon^{0})$
C_F^2	6	34.12	82.98	34.59	-543.8
$C_\text{A}\, C_\text{F}$	2	9.721	1.209	-142.2	-696.6
C_A^2	1	6.497	12.80	15.87	-47.92
$C_\text{F}\, T_\text{R}\, n_\text{f}$	0	$-\frac{13}{3}$	-32.40	-127.9	-355.2
$C_\text{A}\, T_\text{R}\, n_\text{f}$	0	$-\frac{3}{2}$	-12.01	-46.90	-104.1

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)
- higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\mathrm{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\mathrm{Col},i)}(p_1, p_2, p_3) + \mathrm{O}(\epsilon^1)$$

- kinematical point parametrized by y_{ij}

$$y_{12} = 0.238667, \qquad y_{13} = 0.758153, \qquad y_{23} = 0.003180$$

Col	$\mathrm{O}(\epsilon^{-4})$	$\mathrm{O}(\epsilon^{-3})$	$\mathrm{O}(\epsilon^{-2})$	$\mathrm{O}(\epsilon^{-1})$	$\mathrm{O}(\epsilon^{0})$
C_F^2	6	36.79	106.0	120.6	-431.0
$C_\mathrm{A}\, C_\mathrm{F}$	2	25.38	143.6	537.3	1505
C_A^2	1	15.24	119.5	660.5	2902
$C_\mathrm{F}\, T_\mathrm{R}\, n_\mathrm{f}$	0	$-\frac{13}{3}$	-31.30	-121.7	-346.0
$C_\mathrm{A}\, T_\mathrm{R}\, n_\mathrm{f}$	0	$-\frac{3}{2}$	-17.72	-109.1	-470.9

Integrated approximate cross sections - an example

Iterated singly-unresolved

- example: $e^+e^- \to 3$ jets (momentum assignment is $1_q, 2_{\bar{q}}, 3_g$)
- higher order expansion coefficients can be computed numerically

$$\mathbf{I}_{12}^{(0)}(p_1, p_2, p_3; \epsilon) = \left[\frac{\alpha_s}{2\pi} S_\epsilon \left(\frac{\mu^2}{Q^2}\right)^\epsilon\right]^2 \sum_{i=-4}^{0} \sum_{\text{color}} \frac{\text{Col}}{\epsilon^i} \mathcal{I}_{12,3j}^{(\text{Col},i)}(p_1, p_2, p_3) + \mathrm{O}(\epsilon^1)$$

- kinematical point parametrized by y_{ij}

$$y_{12} = 0.937044, \qquad y_{13} = 0.024207, \qquad y_{23} = 0.038749$$

Col	$\mathrm{O}(\epsilon^{-4})$	$\mathrm{O}(\epsilon^{-3})$	$\mathrm{O}(\epsilon^{-2})$	$\mathrm{O}(\epsilon^{-1})$	$\mathrm{O}(\epsilon^{0})$
C_F^2	6	25.85	34.59	-84.25	-566.8
$C_\mathrm{A}\, C_\mathrm{F}$	2	27.79	136.8	330.6	46.20
C_A^2	1	21.02	195.4	1174	5355
$C_\mathrm{F}\, T_\mathrm{R}\, n_\mathrm{f}$	0	$-\frac{13}{3}$	-57.59	-405.2	-2120
$C_\mathrm{A}\, T_\mathrm{R}\, n_\mathrm{f}$	0	$-\frac{3}{2}$	-24.07	-194.7	-1083

Overview

Counterterm	Types of integrals	Done
$\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$	tree level singly-unresolved	✔
$\int_1 \mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$	one-loop singly-unresolved	✔
$\int_1 (\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1})^{\mathrm{A}_1}$	tree level iterated singly-unresolved (1)	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$	tree level iterated singly-unresolved (2)	✔
$\int_2 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$	tree level doubly-unresolved	✔/✘

Outlook

Present status

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1} + \sigma^{\mathrm{NNLO}}_{m}$$

Each integrable in four dimensions

$$\sigma^{\mathrm{NNLO}}_{m+2} = \int_{m+2} \left\{ \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} J_m - \left[\mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} J_m \right] \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m+1} = \int_{m+1} \left\{ \left[\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right] J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] J_m \right\}$$

$$\sigma^{\mathrm{NNLO}}_{m} = \int_{m} \left\{ \mathrm{d}\sigma^{\mathrm{VV}}_{m} + \int_2 \mathrm{d}\sigma^{\mathrm{RR,A_2}}_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A_{12}}}_{m+2} \right] + \int_1 \left[\mathrm{d}\sigma^{\mathrm{RV,A_1}}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR,A_1}}_{m+2} \right)^{\mathrm{A_1}} \right] \bigg\} J_m$$

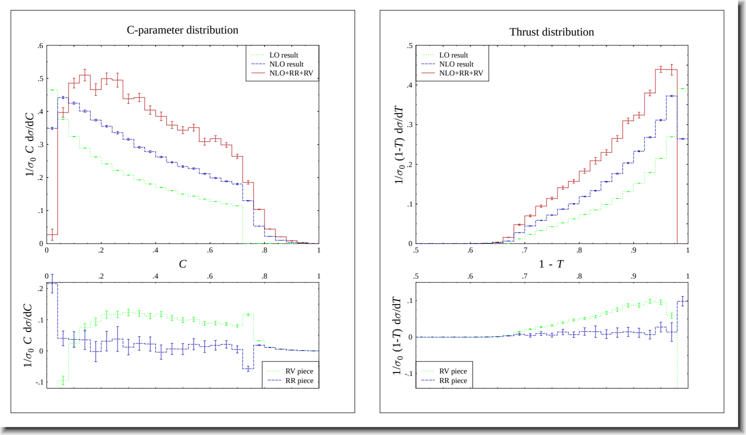
- ✔ unintegrated RR counterterms
- ✔ unintegrated RV counterterms
- ✔ tree-level and one-loop singly-unresolved integrals
- ✔ tree-level iterated singly-unresolved integrals
- ➠ tree-level doubly-unresolved integrals

Present status

NNLO correction is the sum of three terms

$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} = \boxed{\sigma^{\mathrm{NNLO}}_{m+2} + \sigma^{\mathrm{NNLO}}_{m+1}} + \sigma^{\mathrm{NNLO}}_{m}$$

Numerical Monte Carlo integration (single CPU, $\sim$ 50 hours)



Conclusions

Hope to have convinced you

- differential NNLO is interesting and relevant
- subtraction is the method of choice for general, efficient calculations

Local subtraction at NNLO

- general, explicit, local subtraction scheme for computing NNLO jet cross sections, for processes with no colored particles in the initial state
- investigated various methods to compute the integrated counterterms: IBP's, MB, SD
- integration of all singly-unresolved and iterated singly-unresolved counterterms finished
- integration of doubly-unresolved counterterms underway

Next steps

- finish doubly-unresolved integrals: <10 double soft counterterms left
- physical applications
- extend scheme to hadron-initiated and massive processes