Semi-classical approximations based on de Broglie-Bohm theory

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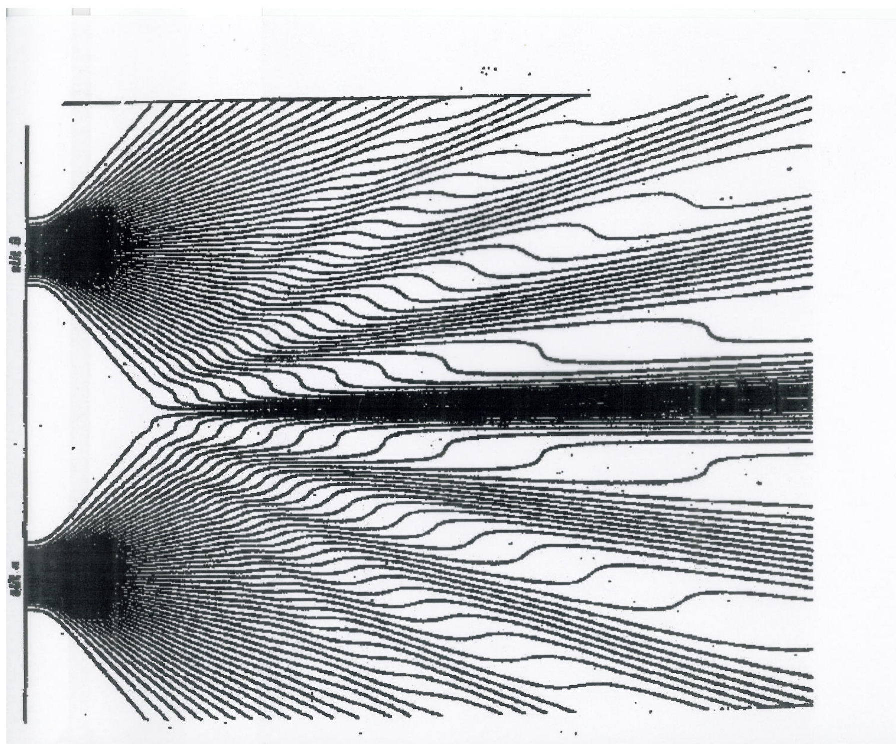

Semi-classical gravity

There are conceptual difficulties with the quantum treatment of gravity: problem of time, finding solutions to Wheeler-DeWitt equation, etc. Therefore one often resorts to semi-classical approximations:

→ **Matter is treated quantum mechanically**, as quantum field on curved space-time.
E.g. scalar field $\Psi(\phi)$

→ **Gratívity is treated classically**, described by

$$G_{\mu\nu} = \frac{8\pi G}{c^4} \langle \Psi | \widehat{T}_{\mu\nu} | \Psi \rangle$$

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In de Broglie-Bohm theory matter is described by $\Psi(\phi)$ and actual scalar field $\phi_B(\mathbf{x}, t)$. Proposal for semi-classical theory:

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Similar situation in scalar electrodynamics:
Quantum matter field described by $\Psi(\phi)$ and actual scalar field $\phi_B(\mathbf{x}, t)$. Semi-classical theory:

$$\partial_\mu F^{\mu\nu} = j^\nu(\phi_B)$$

$\rightarrow$ In general doesn't work because $\partial_\nu j^\nu(\phi_B) \neq 0$!

Outline

- Introduction to de Broglie-Bohm
- Semi-classical approximation in non-relativistic de Broglie-Bohm
- Semi-classical approximation in scalar electrodynamics
- Semi-classical approximation in mini-superspace model

Non-relativistic de Broglie-Bohm theory
(a.k.a. pilot-wave theory, Bohmian mechanics, . . .)

- De Broglie (1927), Bohm (1952)


- Particles moving under influence of the wave function.

- Dynamics:

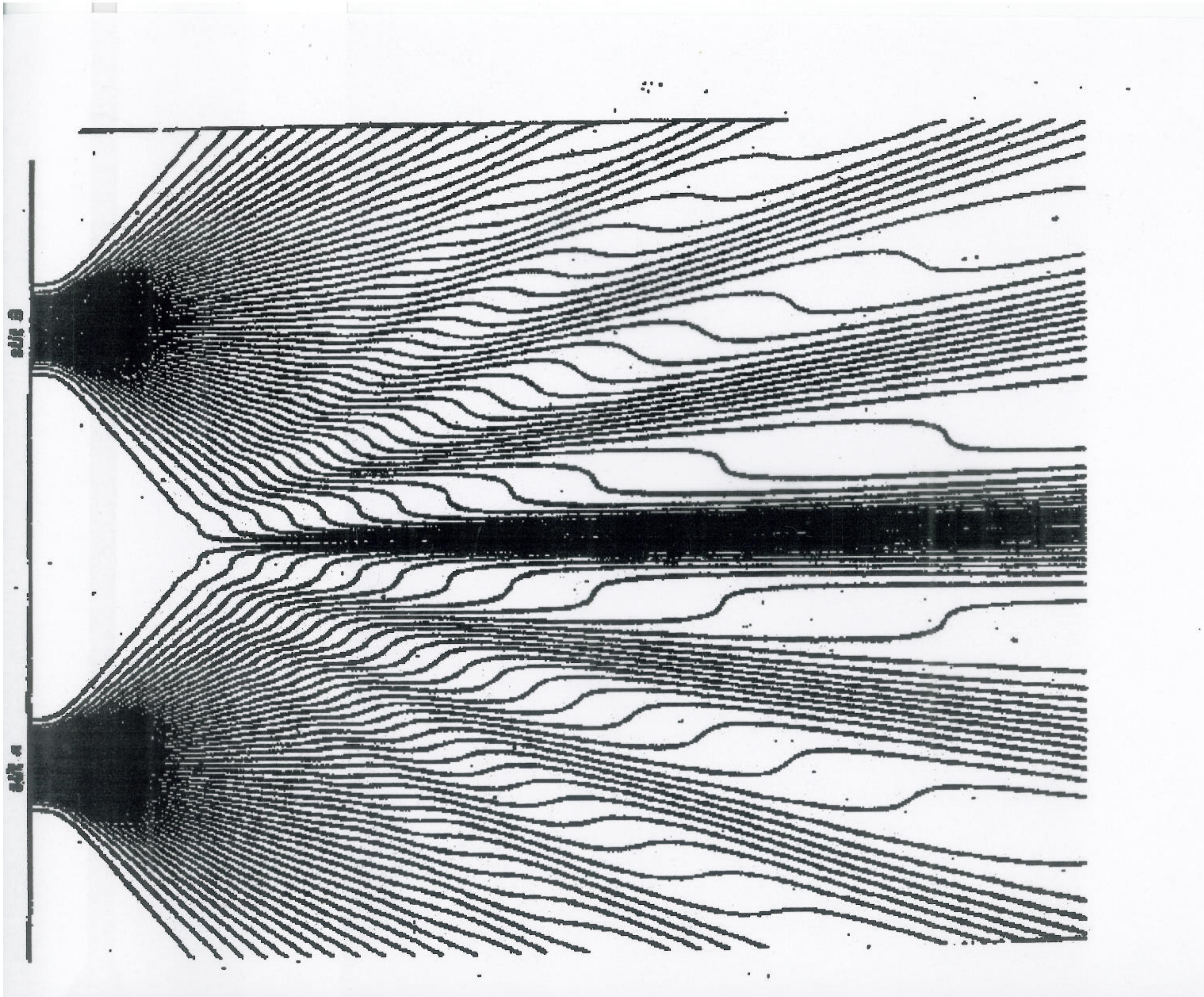
$$\mathrm{i}\hbar\partial_t\psi = \left(-\sum_{k=1}^{n}\frac{\hbar^2}{2m_k}\nabla_k^2 + V\right)\psi$$

$$\frac{d\mathbf{x}_k}{dt} = \mathbf{v}_k^{\psi}(\mathbf{x}_1, \dots, \mathbf{x}_n, t),$$

where

$$\mathbf{v}_k^{\psi} = \frac{\hbar}{m_k}\mathrm{Im}\frac{\boldsymbol{\nabla}_k\psi}{\psi} = \frac{1}{m_k}\boldsymbol{\nabla}_k S, \qquad \psi = |\psi|e^{\mathrm{i}S/\hbar}$$

• Double Slit experiment:



- Quantum equilibrium:

  - for an ensemble of systems with wave function ψ
  - distribution of particle positions $\rho(x) = |\psi(x)|^2$

- Quantum equilibrium is preserved by the particle motion because it satisfies the continuity equation:

$$\partial_t |\psi|^2 + \sum_{k=1}^{n} \boldsymbol{\nabla}_k \cdot (\mathbf{v}_k^{\psi} |\psi|^2) = 0$$

$\rightarrow$ For other Schrödinger equations, the continuity equation of $|\psi|^2$ may be used to find a suitable guidance law.

That is

$$\partial_t |\psi|^2 + \mathrm{div} j^{\psi} = 0$$

suggests the guidance law

$$\dot{X} = \frac{j^{\psi}}{|\psi|^2}$$

(treatment of arbitrary Hamiltonians: Struyve & Valentini (2009))

- **Classical limit:**

$$\dot{\mathbf{x}} = \frac{1}{m}\boldsymbol{\nabla} S \qquad \Rightarrow \qquad m\ddot{\mathbf{x}} = -\boldsymbol{\nabla}(V + Q)$$

$$\psi = |\psi| e^{\mathrm{i}S/\hbar}, \qquad Q = -\frac{\hbar^2}{2m}\frac{\nabla^2|\psi|}{|\psi|} = \text{quantum potential}$$

Classical trajectories when $|\boldsymbol{\nabla} Q| \ll |\boldsymbol{\nabla} V|$.

- **Wave function of subsystem: conditional wave function**

Consider composite system: $\psi(x_1, x_2, t)$, $(X_1(t), X_2(t))$

Conditional wave function for system 1:

$$\chi(x,t) = \psi(x_1, X_2(t), t)$$

The trajectory $X_1(t)$ satisfies

$$\frac{dX_1(t)}{dt} = v^{\chi}(X_1(t), t) = \frac{1}{m_1}\mathrm{Im}\frac{\nabla_1 \chi(x_1,t)}{\chi(x_1,t)}\bigg|_{x_1 = X_1(t)}$$

Conditional wave function χ:

- will satisfy Schrödinger equation in certain cases

- will undergo collapse

Semi-classical approximation: non-relativistic quantum mechanics

- System 1: quantum mechanical. System 2: classical

  Usual approach (mean field):

$$\mathrm{i}\partial_t \psi(x_1, t) = \left(-\frac{\nabla_1^2}{2m_1} + V(x_1, X_2(t)) \right) \psi(x_1, t)$$

$$m_2 \ddot{X}_2(t) = \langle \psi | F_2(x_1, X_2(t)) | \psi \rangle \ , \qquad F_2 = -\nabla_2 V$$

$\rightarrow$ **backreaction through mean force**

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→ **backreaction through mean force**

de Broglie-Bohm-based approach:

$$\mathrm{i}\partial_t \psi(x_1,t) = \left(-\frac{\nabla_1^2}{2m_1} + V(x_1, X_2(t)) \right) \psi(x_1,t)$$

$$\dot{X}_1(t) = v_1^{\psi}(X_1(t),t)\,, \qquad m_2 \ddot{X}_2(t) = F_2(X_1(t), X_2(t))$$

→ **backreaction through Bohmian particle**

- Prezhdo and Brookby (2001):

  De Broglie-Bohm-based approach yields better results than usual approach:

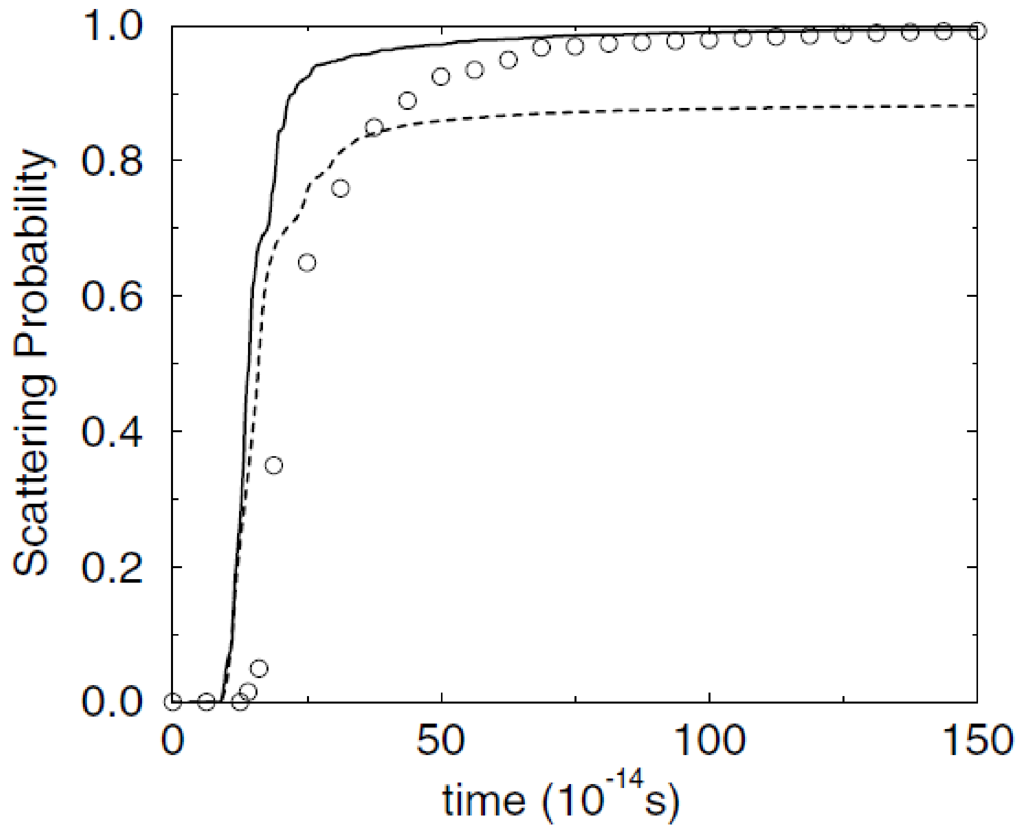

FIG. 1. The time-dependent scattering probability P_s, Eq. (16), for the model problem detailed in the text obtained for the incident energy of 20 kJ/mol using exact quantum dynamics (circles), mean-field dynamics (dashed curve), and the Bohmian quantum-classical technique (solid curve).

- **Derivation of de Broglie-Bohm-based semi-classical approximation**

Full quantum mechanical description:

$$\mathrm{i}\partial_t \psi(x_1,x_2,t) = \left(-\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} + V(x_1,x_2)\right)\psi(x_1,x_2,t)$$

$$\dot{X}_1(t) = v_1^{\psi}(X_1(t),X_2(t),t)\,, \quad \dot{X}_2(t) = v_2^{\psi}(X_1(t),X_2(t),t)$$

Conditional wave function $\chi(x_1,t) = \psi(x_1,X_2(t),t)$ satisfies

$$\mathrm{i}\partial_t \chi(x_1,t) = \left(-\frac{\nabla_1^2}{2m_1} + V(x_1,X_2(t))\right)\chi(x_1,t) + I(x_1,t)$$

and particle two:

$$m_2\ddot{X}_2(t) = -\nabla_2 V(X_1(t),x_2)\Big|_{x_2=X_2(t)} - \nabla_2 Q(X_1(t),x_2)\Big|_{x_2=X_2(t)}$$

$\rightarrow$ Semi-classical approximation follows when I and $-\nabla_2 Q$ are negligible (e.g. when particle 2 is much heavier than particle 1)

Semi-classical approximation: scalar electrodynamics

- Classical field equations:

$$D_\mu D^\mu \phi + m^2 \phi = 0\,, \qquad \partial_\mu F^{\mu\nu} = j^\nu = \mathrm{i}e\left(\phi^* D^\nu \phi - \phi D^{\nu *} \phi^*\right)$$

with $D_\mu = \partial_\mu + \mathrm{i}eA_\mu$, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and j^μ charge current ($\partial_\mu j^\mu = 0$).

- Quantum field theory in Coulomb gauge:

$$\ldots$$

- De Broglie-Bohm approach:
  - Wave functional: $\Psi(\phi, \mathbf{A}^T, t)$
  - Actual field configurations: $\phi(x), \mathbf{A}^T(x)$

$$\ldots$$

- De Broglie-Bohm-based semi-classical approximation:
  - Matter field: quantum mechanical; Electromagnetic field: classical
  - Schrödinger equation for $\Psi(\phi, t)$:

$$\mathrm{i}\partial_t\Psi = \int d^3x \left(-\frac{\delta^2}{\delta\phi^*\delta\phi} + |(\boldsymbol{\nabla} - \mathrm{i}e\mathbf{A}^T)\phi|^2 + m^2|\phi|^2 - \frac{1}{2}\mathcal{C}\frac{1}{\nabla^2}\mathcal{C} \right)\Psi\,,$$

where $\mathcal{C}$ is the charge density operator.
  - Guidance equation scalar field:

$$\dot{\phi} = \frac{\delta S}{\delta\phi^*} - e\phi\frac{1}{\nabla^2}\mathcal{C}S \qquad \Rightarrow \qquad D_\mu D^\mu\phi - m^2\phi = -\frac{\delta Q}{\delta\phi^*}$$

  - Classical Maxwell equations with quantum correction:

$$\partial_\mu F^{\mu\nu} = j^\nu + j^\nu_Q\,,$$

with "quantum charge current" $j^\nu_Q = (0, \mathbf{j}_Q)$: $\qquad \mathbf{j}_Q = \mathrm{i}\boldsymbol{\nabla}\frac{1}{\nabla^2}\mathcal{C}Q$

Is consistent since: $\partial_\mu(j^\mu + j^\mu_Q) = 0$**.**

$\rightarrow$ **Crucial in the derivation was that gauge was eliminated!**
How to eliminate it in canonical quantum gravity?
(in this case: gauge = spatial diffeomorphism invariance).

Semi-classical approximation: mini-superspace model

- Restriction to homogeneous and isotropic (FRW) metrics and fields:
  - Gravity: $\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\Omega_3^2$
  - Matter: $\phi = \phi(t)$

  Wheeler-DeWitt equation:

$$(H_G + H_M)\psi = 0\,,$$

$$H_G = \frac{1}{4a^2}\partial_a(a\partial_a) + a^3V_G\,, \qquad H_M = -\frac{1}{2a^3}\partial_\phi^2 + a^3V_M$$

  Guidance equations ($N = 1$):

$$\dot{a} = -\frac{1}{2a}\partial_a S\,, \qquad \dot{\phi} = \frac{1}{a^3}\partial_\phi S \tag{1}$$

- Semi-classical approximation:

$$\mathrm{i}\partial_t\psi = H_M\psi\,, \qquad \dot{\phi} = \frac{1}{a^3}\partial_\phi S \tag{2}$$

  and Friedmann equation with quantum correction:

$$\frac{\dot{a}^2}{a^2} = \frac{\dot{\phi}^2}{2} + V_M + V_G + Q \tag{3}$$

Summary

Results so far:

- Consistent semi-classical approximation for:
  - Non-relativistic systems
  - Quantum electrodynamics
  - Mini-superspace models

To do:

- Find semi-classical approximation for full quantum gravity
- Find higher order correction terms
- Develop rigorous expansions (e.g. à la WKB)
- Find applications (e.g. in cosmology: Hawking radiation, cosmological perturbations in inflation theory)