(Recent advances in)

Overlap integrals, spectroscopic factors and asymptotic normalization coefficients for one-nucleon transfer reactions

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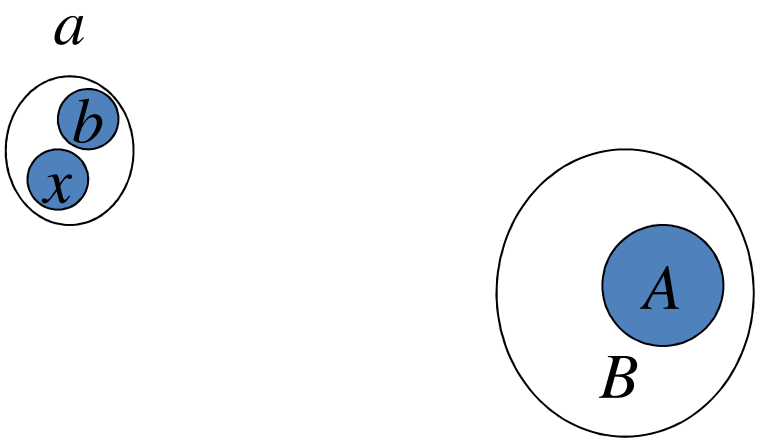
Transfer reaction

$$A + a \rightarrow B + b \quad (\alpha \rightarrow \beta)$$



Amplitude for transfer reactions A(a,b)B

$$T_{\beta\alpha}^{\mathrm{DW}}(\mathbf{k}_\beta, \mathbf{k}_\alpha) = \langle \chi_\beta^{(-)}(\mathbf{k}_\beta)\,\psi_\beta | W | \chi_\alpha^{(+)}(\mathbf{k}_\alpha)\,\psi_\alpha \rangle$$

$$= \iint d\mathbf{r}_\beta\, d\mathbf{r}_\alpha\, \chi_\beta^{(-)}(\mathbf{k}_\beta, \mathbf{r}_\beta)^*\, (\psi_\beta | W | \psi_\alpha)\, \chi_\alpha^{(+)}(\mathbf{k}_\alpha, \mathbf{r}_\alpha)$$

$$= \iint d\mathbf{r}_\beta\, d\mathbf{r}_\alpha\, \chi_\beta^{(-)}(\mathbf{k}_\beta, \mathbf{r}_\beta)^*\, \mathrm{I}_{\beta\alpha}(\mathbf{r}_\beta, \mathbf{r}_\alpha)\, \chi_\alpha^{(+)}(\mathbf{k}_\alpha, \mathbf{r}_\alpha).$$

$$I_{\beta\alpha}(\mathbf{r}_\beta, r_\alpha) = (\psi_B \psi_b | W | \psi_A \psi_a)$$

Overlap integrals

Overlap integrals $\langle \psi_B | \psi_A \rangle$ carry information about nuclear structure. They are solutions of an integral equation.

$$(T_A + V_A - E_A)\psi_A = 0, \qquad (T_B + V_B - E_B)\psi_B = 0$$

$$\psi_A (T_B + V_B - E_B)\, \psi_B = 0$$

$$\langle \psi_A | T_A + (T_B - T_A) + V_A + (V_B - V_A) - E_A + (E_A - E_B) | \psi_B \rangle = 0$$

$$\langle \psi_A | (T_B - T_A) + (E_A - E_B) | \psi_B \rangle = \langle \psi_A | (V_A - V_B) | \psi_B \rangle$$

$$(T_x + \varepsilon)\langle \psi_A | \psi_B \rangle = -\langle \psi_A | V_{Ax} | \psi_B \rangle$$

Partial wave expansion of the overlap integral

$$I_{AB}(\boldsymbol{r}) \equiv \left\langle \psi_B \middle| \psi_A \right\rangle = \sum_{M_A M_B m \sigma} (lm\, ½\sigma \mid jm_j)(jm_j J_A M_A \mid J_B M_B) I_{lj}(r) Y_{lm}(\hat{r}) \chi_{1/2\sigma} \chi_{1/2\tau}$$

Properties of the overlap integrals

I) Asymptotic behaviour

At large r the overlap integral satisfies the equation

$$(T_x + \varepsilon) I_{AB}(\boldsymbol{r}) = -\langle \psi_A | V_{Ax} | \psi_B \rangle \approx 0 \qquad (\textit{for neutral particle } x)$$

$$(T_x + V_{coul}(r) + \varepsilon) I_{AB}(\boldsymbol{r}) = -\langle \psi_A | V_{Ax} - V_{Coul}(r) | \psi_B \rangle \approx 0 \qquad (\textit{for charged particle } x)$$

The asymptotic part of the overlap functions $I_{lj}(r)$ is given by

$$I_{lj}(r) \approx C_{lj}\, W_{-\eta, l+\frac{1}{2}}(2\kappa r)/r$$

C_{lj} is the **<u>asymptotic normalization coefficient</u> (*ANC*),**
W is the Whittaker function,
$\kappa = (2\mu\varepsilon)^{1/2}$, ε is the nucleon separation energy

Example: for $B = A + neutron$ and $l = 0$: $\quad I_{lj}(r) \approx C_{lj}\, exp(-\kappa r)/r$

II) Normalization

Definition: the norm of $I_{lj}(r)$ is called the **spectroscopic factor.**

$$S_{lj} = \int_0^\infty dr\, r^2 I_{lj}^2(r) \qquad (\times B)$$

The meaning of the spectroscopic factor from the shell model point of view.
The shell model wave function is a linear combination of the Slater determinants

$$\psi_A = \sum_{\alpha_A} C_{A,\alpha_A} D_{A,\alpha_A} \qquad \alpha_A = \{n_1 l_1 j_1 m_1 \tau_1, ..., n_A l_A j_A m_A \tau_A\}$$

$$\psi_B = \sum_{\alpha_B} C_{B,\alpha_B} D_{B,\alpha_B}$$

$$\langle \psi_A | \psi_B \rangle = \sum_{\alpha_A \alpha_B} C_{A,\alpha_A} C_{B,\alpha_B} \varphi_{\gamma(\alpha_A \alpha_B)}$$

The spectroscopic factor is expressed only via coefficients C_{A,α_A} and C_{B,α_B} which are probability amplitudes of a particular shell occupation scheme.

Modelling the overlap functions:

$$(T_x + \varepsilon)\langle \psi_A | \psi_B \rangle = -\langle \psi_A | V_{Ax} | \psi_B \rangle \approx -V_{Ax} \langle \psi_A | \psi_B \rangle$$

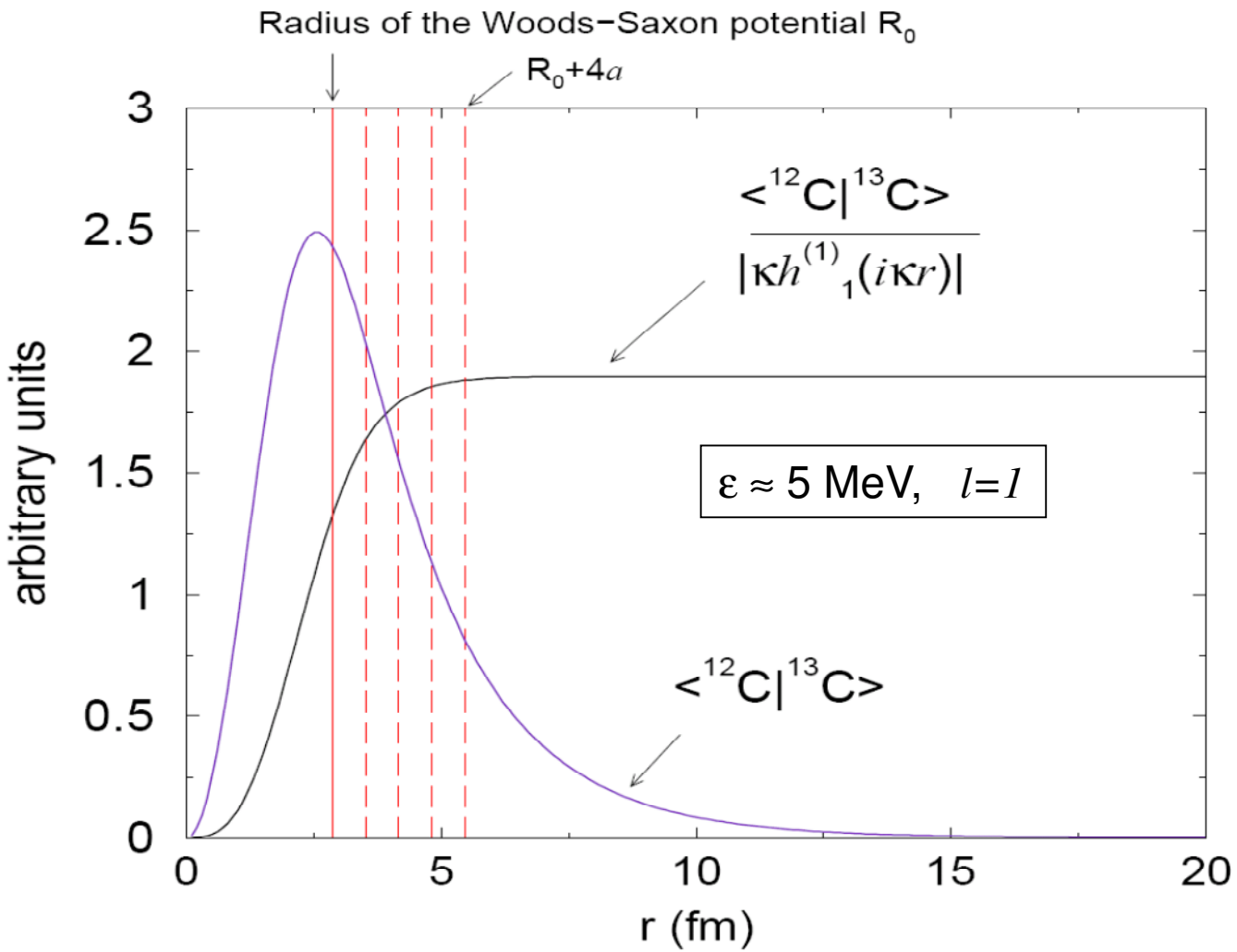
or $$(T_x + V_{Ax}(r) + \varepsilon)\, I_{lj}(r) = 0$$

$$I_{lj}(r) = S^{1/2}\, \varphi_{lj}(r), \qquad \int_0^\infty dr\, r^2 \varphi_{lj}^2(r) = 1$$

$\varphi_{lj}(r)$ is the normalized solution of the two-body equation and the spectroscopic factor S is thought to be determined from experiment.

Often, a standard Wood-Saxon potential with $r_0 \approx$1.25 fm, $a \approx$ 0.65 fm is used to determine $\varphi_{lj}(r)$ while the depth V_0 is fitted to reproduce ε.

Typical example of the overlap functions for stable nuclei



Effective potentials from ab-initio GFMC calculations for overlaps

I. Brida et al, Phys. Rev. C 84, 024319 (2011)

Parent $^{A}Z(J^{\pi}, T)$	Core $^{A-1}Z(J^{\pi}, T)$	l_j	V_{WS} (MeV)	R_{WS} (fm)	a_{WS} (fm)
$^{3}H(\frac{1}{2}^{+}, \frac{1}{2})$	$^{2}H(1^{+}, 0)$	$s_{1/2}$	−172.88	0.56	0.69
		$d_{3/2}$	−2732.90	−1.15	0.91
$^{3}He(\frac{1}{2}^{+}, \frac{1}{2})$	$^{2}H(1^{+}, 0)$	$s_{1/2}$	−179.94	0.54	0.68
		$d_{3/2}$	−8155.10	−2.19	0.91
$^{4}He(0^{+}, 0)$	$^{3}H(\frac{1}{2}^{+}, \frac{1}{2})$	$s_{1/2}$	−202.21	0.93	0.66
	$^{3}He(\frac{1}{2}^{+}, \frac{1}{2})$	$s_{1/2}$	−200.93	0.88	0.69
$^{7}Li(\frac{3}{2}^{-}, \frac{1}{2})$	$^{6}He(0^{+}, 1)$	$p_{3/2}$	−58.93	2.68	0.93
	$^{6}Li(1^{+}, 0)$	$p_{1/2}$	−41.80	3.18	0.85
		$p_{3/2}$	−69.55	1.89	1.17
	$^{6}Li(3^{+}, 0)$	$p_{3/2}$	−62.98	2.35	1.18
	$^{6}Li(0^{+}, 1)$	$p_{3/2}$	−59.39	2.64	0.97
$^{7}Li(\frac{1}{2}^{-}, \frac{1}{2})$	$^{6}Li(1^{+}, 0)$	$p_{1/2}$	−33.71	3.39	0.31
		$p_{3/2}$	−65.00	2.04	1.15
$^{7}Be(\frac{3}{2}^{-}, \frac{1}{2})$	$^{6}Li(1^{+}, 0)$	$p_{1/2}$	−39.45	3.32	0.76
		$p_{3/2}$	−72.22	1.85	1.11
	$^{6}Li(3^{+}, 0)$	$p_{3/2}$	−59.20	2.52	1.07
	$^{6}Li(0^{+}, 1)$	$p_{3/2}$	59.49	2.64	0.96

R_{st}=2.27

Source term approach:

source term

$$(T_l + \frac{Z_{A-1}e^2}{r} + \varepsilon) I_{lj}(r) = U_{lj}(r) \qquad U_{lj}(r) = \left\langle [[Y_l(\hat{r}) \otimes \chi_{1/2}]_j \otimes \Psi_{J_{A-1}}]_{J_A} \middle\| \hat{V} \middle\| \Psi_{J_A} \right\rangle$$

Wave function Ψ_{A-1} and Ψ_A are replaced by model wave functions Φ_A and Φ_B taken from the $0\hbar\omega$ oscillator shell model (which for closed shell model are the same as in the IPM)

Effective interaction V:

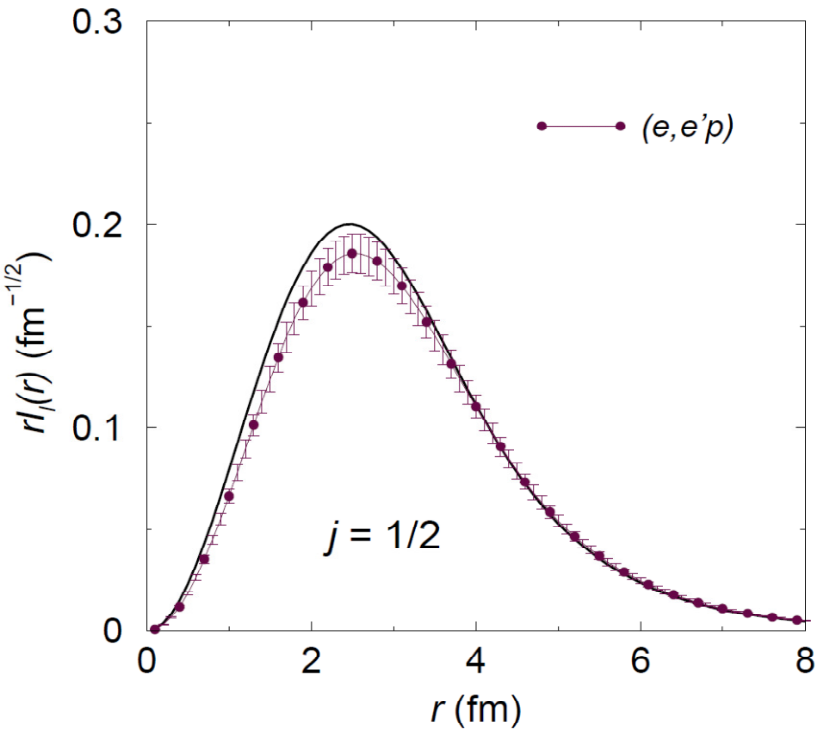
$$\hat{V} = \sum_{i=1}^{A-1} V_{NN}\left(|r_i - r_A|\right) + \sum_{i=1}^{A-1} \frac{e_i e_A}{|r_i - r_A|} - \frac{Z_{A-1} e_A e}{r}$$

For the two-body NN potential the M3YE potential is used from
Bertsch et al, Nucl. Phys. A 284 (1977) 399

$V_{ST} = V_{1,ST} \exp(-a_{1,ST} r)/r + V_{2,ST} \exp(-a_{2,ST} r)/r + V_{3,ST} \exp(-a_{3,ST} r)/r$ + *spin-orbit+tensor…*

Coefficients $V_{i,ST}$ and $a_{i,ST}$ have been found by fitting the matrix elements derived from the NN elastic scattering data (*Elliot et al, NPA121 (1968) 241)*

$<^{16}O|^{15}N>$



S_{STA} = 1.45

experiment:		*Shell model*		*Reduction factor*
(*e,e'p*)	1.27 ± 0.13	0ħω (non TI)	2.0	0.64 ± 0.07
p knockout	1.12 ± 0.07	0ħω (TI)	2.13	
(*p,d*)	1.48 ± 0.16	4ħω (non TI)	1.65	

A	$A-1$	S^{DO}	S^{IE}	S_{exp}	*experiment*	S_{VMC}	S_{GFMC}
^{7}Li	^{6}He	0.69	0.28	0.42(4)	(e,e'p)	0.42	0.41
^{7}Li	^{6}Li	0.87	0.44	0.74(11)	(d,t)	0.68	0.67
^{8}Li	^{7}He	1.02	0.38	0.36(7)	(d,3He)	0.58	
^{8}Li	^{7}Li	1.14	0.78			0.97	
^{8}B	^{7}Be	1.14	0.78	0.89(7)	p knockout	0.97	
^{9}Li	^{8}Li	1.04	0.60	0.59(15)	(d,t)	1.14	
^{9}Be	^{8}Li	1.13	0.45			0.73	
^{9}C	^{8}B	1.04	0.71	0.77(6)	p knockout	1.14	
^{10}Be	^{9}Li	1.93	0.81			1.04	
^{10}Be	^{9}Be	2.67	1.48			1.93	
^{12}B	^{11}B	0.99	0.97	0.40(6)	(d,p)		
^{12}C	^{11}B	2.85	1.55	1.72(11)	(e,e'p)		
^{13}C	^{12}C	0.63	0.63	0.54(8)	(d,p)		
^{14}C	^{13}C	1.87	1.82	1.07(22)	(d,p)		
^{14}N	^{13}N	0.72	0.60	0.48(8)	(p,d)		
^{15}N	^{14}N	1.48	1.31	0.93(15)	(d,p)		
^{16}O	^{15}N	2.13	1.45	1.27(13)	(e,e'p)		

A ≥ 16 nuclei

SF of double-closed shell nuclei obtained from STA calculations:
Oscillator IPM wave functions are used with $\hbar\omega = 41A^{-1/3} - 25A^{-2/3}$
and the M3YE (central + spin-orbit) NN potential

A	A-1	lj	S_{IPM}	S_{exp}	S_{STA}	S_{STA}/S_{IPM}
^{16}O	^{15}N	$p_{1/2}$	2.0	1.27(13)	1.45	0.73
		$p_{3/2}$	4.0	2.25(22)	2.61	0.65
^{40}Ca	^{39}K	$d_{3/2}$	4.0	2.58(19)	2.90	0.73
		$s_{1/2}$	2.0	1.03(7)	1.15	0.58
^{48}Ca	^{47}K	$s_{1/2}$	2.0	1.07(7)	1.38	0.69
		$d_{3/2}$	4.0	2.26(16)	2.70	0.68
		$d_{5/2}$	6.0	0.683(49)	4.21	0.71
^{208}Pb	^{207}Tl	$s_{1/2}$	2.0	0.98(9)	1.48	0.74
		$d_{3/2}$	4.0	2.31(22)	2.88	0.72
		$d_{5/2}$	6.0	2.93(28)	4.38	0.73
		$g_{7/2}$	8.0	2.06(20)	4.88	0.61

Comparison to other theoretical calculations

A	A-1	lj	$S_{exp}/(2j+1)$	STA	CBFM correlated basis functions method	SCGFM self-consistent Green's function method	CCM coupled-clusters method
^{16}O	^{15}N	$p_{1/2}$	0.64± 0.07	0.73	0.89	0.8	0.9
		$p_{3/2}$	0.56± 0.06	0.65	0.89	0.8	0.9
^{40}Ca	^{39}K	$d_{3/2}$	0.65±0.05	0.76	0.85	0.8	
		$s_{1/2}$	0.52±0.04	0.58	0.87	0.8	
^{48}Ca	^{47}K	$s_{1/2}$	0.54±0.04	0.69	0.84	0.36	
		$d_{3/2}$	0.57±0.04	0.68	0.86	0.59	
		$d_{5/2}$	0.11±0.02	0.71	0.85		
^{208}Pb	^{207}Tl	$s_{1/2}$	0.49±0.74	0.74	0.85		
		$d_{3/2}$	0.58±0.06	0.72	0.83		
		$d_{5/2}$	0.49±0.05	0.73	0.83		
		$g_{7/2}$	0.26±0.03	0.61	0.82		
		$h_{11/2}$	0.57±0.06	0.48	0.82		

Shell closure away from beta-stability

New magic nucleus: ^{24}O (*C.R.Hoffman et al, Phys.Lett. B 672, 17 (2009)*)
Neutrons occupy shells: $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$, $0d_{5/2}$, $1s_{1/2}$
Protons occupy shells: $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$

One-Neutron Removal Measurement ^{12}C(^{24}O, ^{23}O), E=920 MeV/A
(*R.Kanungo et al, Phys.Rev.Lett. 102, 152501 (2009*))

$S_{exp} = 1.74 \pm 0.19$ for $s_{1/2}$ neutron removal

$S_{IPM} = 2.0$ (or $S = 2.18$ with centre-of mass removal)

S_{SM}(SDPF-M) = 1.769; S_{SM}(USDB) = 1.810

Ab-initio coupled-cluster calculations give $S_{CCM} = 1.83\text{-}1.84$ [*PRC83, 021305*]

Source term approach with oscillator IPM wave functions for ^{24}O and ^{23}O gives

$S_{STA} = 1.66$

For $0p_{1/2}$ ***proton removal*** from ^{24}O $S_{STA} = 1.18$ (as compared to $S_{IPM} = 2$)

CCM calculations give $S_{CCM} = 1.21\text{-}1.30$ [*PRC83, 021305*]

Double magic N=Z nucleus: ^{56}Ni

Fully occupied shells: $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$, $0d_{5/2}$, $1s_{1/2}$, $0d_{3/2}$, $0f_{7/2}$

^{57}Ni has one valence neutron above double closed shell core ^{56}Ni

One-Neutron Removal Measurement ^{9}Be(^{57}Ni,^{56}Ni+γ)X
(*K. L. Yurkewicz et al, Phys.Rev. C **74**, 024304 (2006)*)

$S_{IPM} = 1.0$

$S_{exp} = 0.58 \pm 0.11$ for $p_{3/2}$ removal

Source term approach with oscillator IPM wave functions for ^{57}Ni and ^{56}Ni gives

$S_{STA} = 0.59$

SCGFM gives 0.65 (*C.Barbieri and M.Hjorth-Jensen*, Phys.Rev. C 79, 064313 (2009)

Double magic ^{132}Sn

Fully occupied shells:
Neutrons: $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$, $0d_{5/2}$, $1s_{1/2}$, $0d_{3/2}$, $0f_{7/2}$, $1p_{3/2}$, $0f_{5/2}$, $1p_{1/2}$, $0g_{9/2}$, $0g_{7/2}$, $1d_{5/2}$, $1d_{3/2}$, $2s_{1/2}$, $0h_{11/2}$
Protons: $0s_{1/2}$, $0p_{3/2}$, $0p_{1/2}$, $0d_{5/2}$, $1s_{1/2}$, $0d_{3/2}$, $0f_{7/2}$, $1p_{3/2}$, $0f_{5/2}$, $1p_{1/2}$, $0g_{9/2}$

Final nucleus	J^π	E_x *(MeV)*	S_{STA}/S_{IPM}
^{131}Sn	$3/2^+$	g.s.	0.80
	$1/2^+$	0.332	0.83
	$5/2^+$	1.655	0.81
	$7/2^+$	2.434	0.75
^{131}In	$9/2^+$	g.s.	0.64
	1/2+	0.30	0.74
	3/2+	1.29	0.74

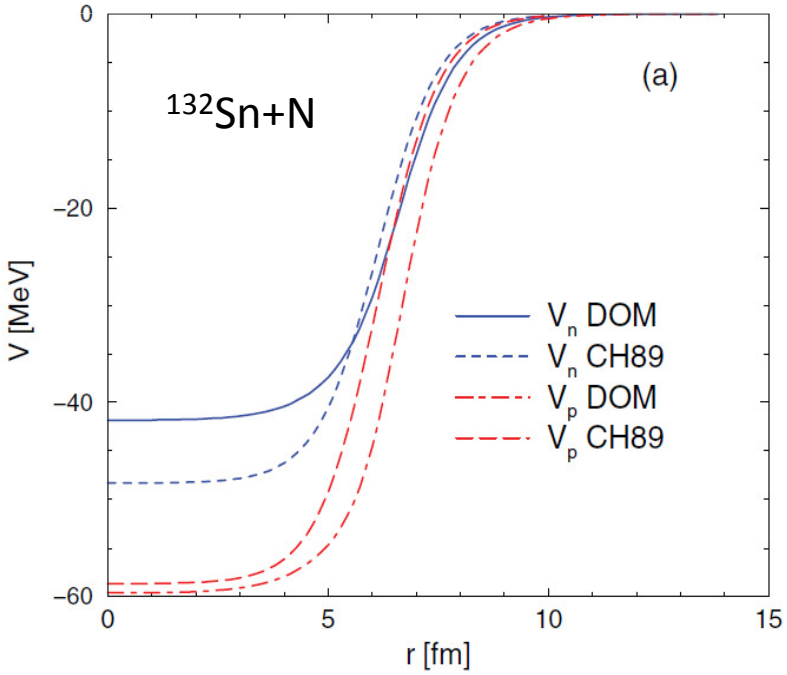
Transfer reactions with dispersive optical potentials

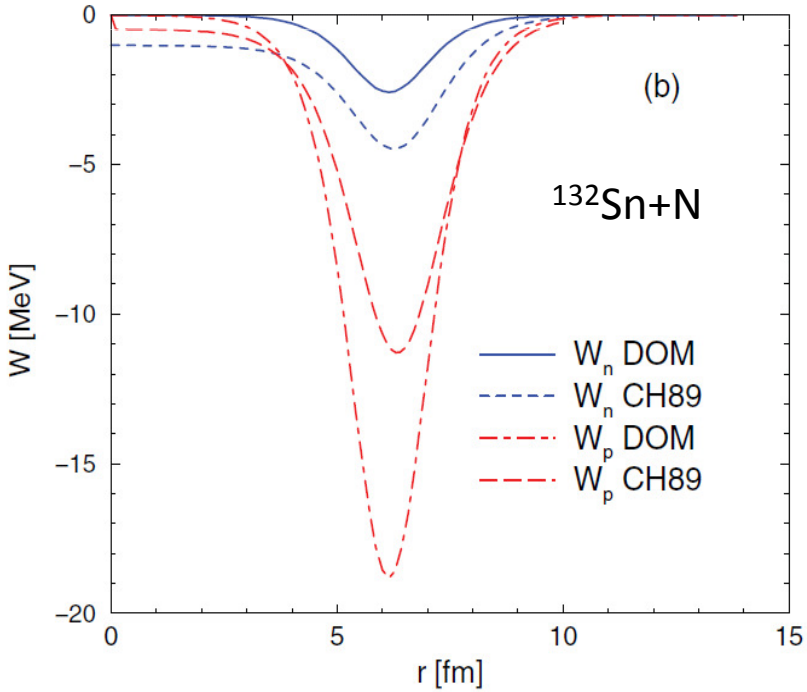
N. B. Nguyen et al, Phys. Rev. C84, 044611 (2011)

$$V_{opt}(\boldsymbol{r},\boldsymbol{r}',E)=V_0(\boldsymbol{r},\boldsymbol{r}')+\Delta V(\boldsymbol{r},\boldsymbol{r}',E)+iW(\boldsymbol{r},\boldsymbol{r}',E)$$

$$\Delta V(\boldsymbol{r},\boldsymbol{r}',E)=\frac{\mathcal{P}}{\pi}\int dE'\ \frac{W(\boldsymbol{r},\boldsymbol{r}',E')}{E-E'}$$

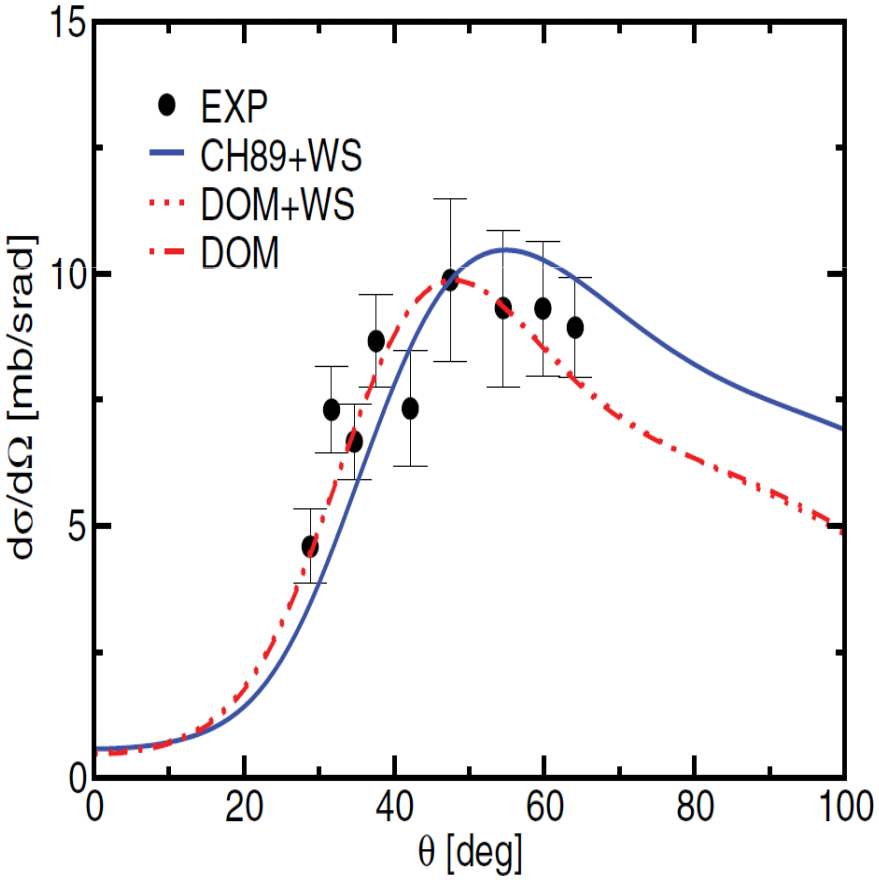
DOM from has been described in terms of 32 parameters used to fit data sets for 40 ≤A ≤ 208 and 4 ≤ E ≤200 MeV (taken from *J.M. Mueller et al, Phys. Rev. C **83**, 064605 (2011))*



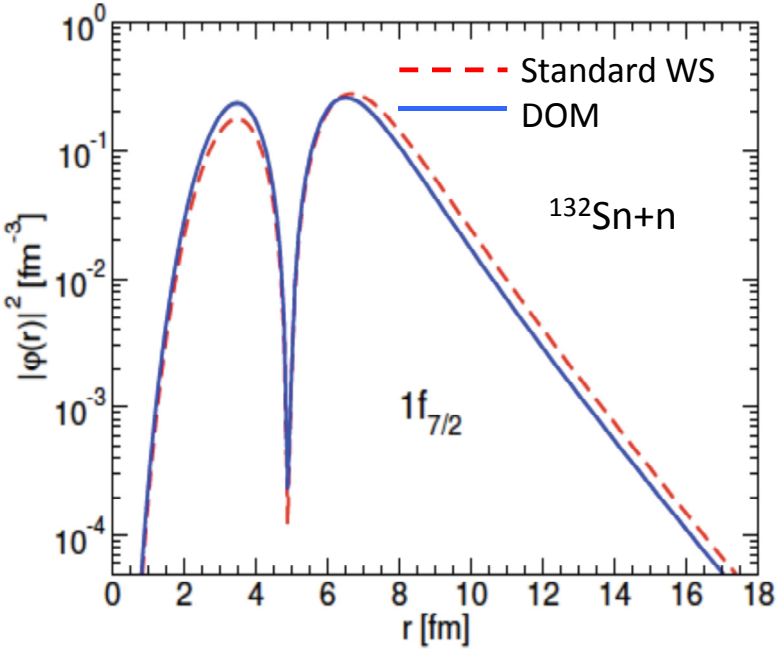


$^{132}Sn(d,p)^{133}Sn$, E_d = 9 .46 MeV

Johnson-Tandy adiabatic model has been used to calculate transfer cross sections, remnant term is neglected.



DOM can also predict potential well for neutron bound state



ANCs obtained from transfer reactions using

- Global systematic of nucleon optical potentials CH89
- DOM

		Woods-Saxon potential used for neutron bound state		DOM used for neutron bound state		
Nucleus	E_d (MeV)	CH89 + WS	DOM + WS	DOM	DOM(th)	STA
^{41}Ca	20	5.0	4.4	4.4	2.8	4.40
	56	4.6	3.8	3.8		
^{49}Ca	2	31.7	24.4	24.4	29.6	15.1
	13	27.9	22.7	22.6		
	19.3	26.0	23.1	23.0		
	56	35.8	23.5	23.2		
^{133}Sn	9.46	0.78	0.71	0.49	0.56	0.42
^{209}Pb	8	4.5	4.1	4.2	2.5	1.97
	20	2.4	1.7	1.7		

Spectroscopic factors obtained using

- Global systematic of nucleon optical potentials CH89
- DOM

Woods-Saxon potential used for neutron bound state

DOM used for neutron bound state

Nucleus	E_d	Data	CH89 + WS	DOM + WS	DOM	DOM(th)	STA
^{41}Ca	20	[29]	0.96	0.85	0.86	0.75	**0.52**
	56	[30]	0.88	0.73	0.74		
^{49}Ca	2	[31]	0.94	0.72	0.66	0.80	**0.67**
	13	[32]	0.82	0.67	0.61		
	19.3	[32]	0.77	0.68	0.62		
	56	[33]	1.1	0.70	0.62		
^{133}Sn	9.46	[1]	1.1	1.0	0.72	0.80	**0.68**
^{209}Pb	8	[34]	1.7	1.5	1.2	0.76	**0.64**
	20	[35]	0.89	0.61	0.51		

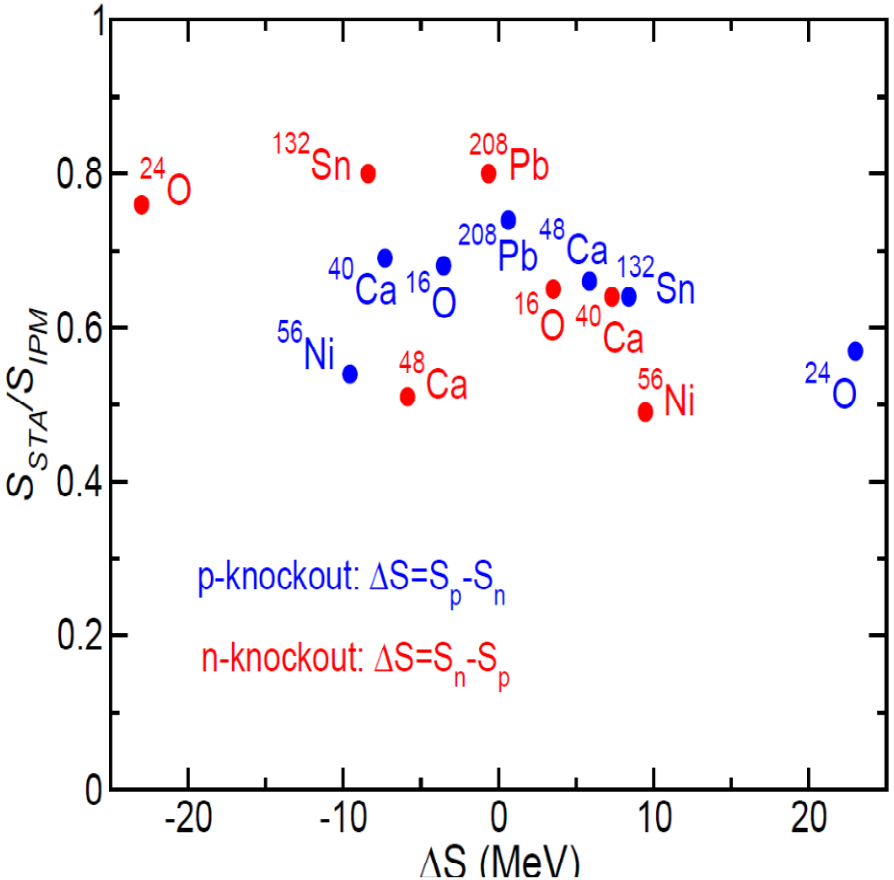
TABLE IV. The ANCs squared C^2_{lj} (in fm^{-1}) for the $\langle^{208}\mathrm{Pb}|^{209}\mathrm{Pb}\rangle$ overlap calculated for several excited states in ^{209}Pb in the STA in comparison to the C^2_{exp} values obtained from periheral transfer reactions in Ref. [27].

E_x	lj	C^2_{STA}	C^2_{exp}	$S_{\mathrm{STA}}/S_{\mathrm{IPM}}$
0	$g_{9/2}$	1.97	2.25 ± 0.11	0.62
0.78	$i_{11/2}$	3.12×10^{-3}	$(1.56 \pm 0.02) \times 10^{-3}$	0.61
1.57	$d_{5/2}$	3.14	10.33 ± 0.49	0.75
2.09	$s_{1/2}$	1.02	36.1 ± 2.0	1.10
2.49	$g_{7/2}$	0.011	0.016 ± 0.001	0.63
2.54	$d_{3/2}$	0.37	2.02 ± 0.12	0.77

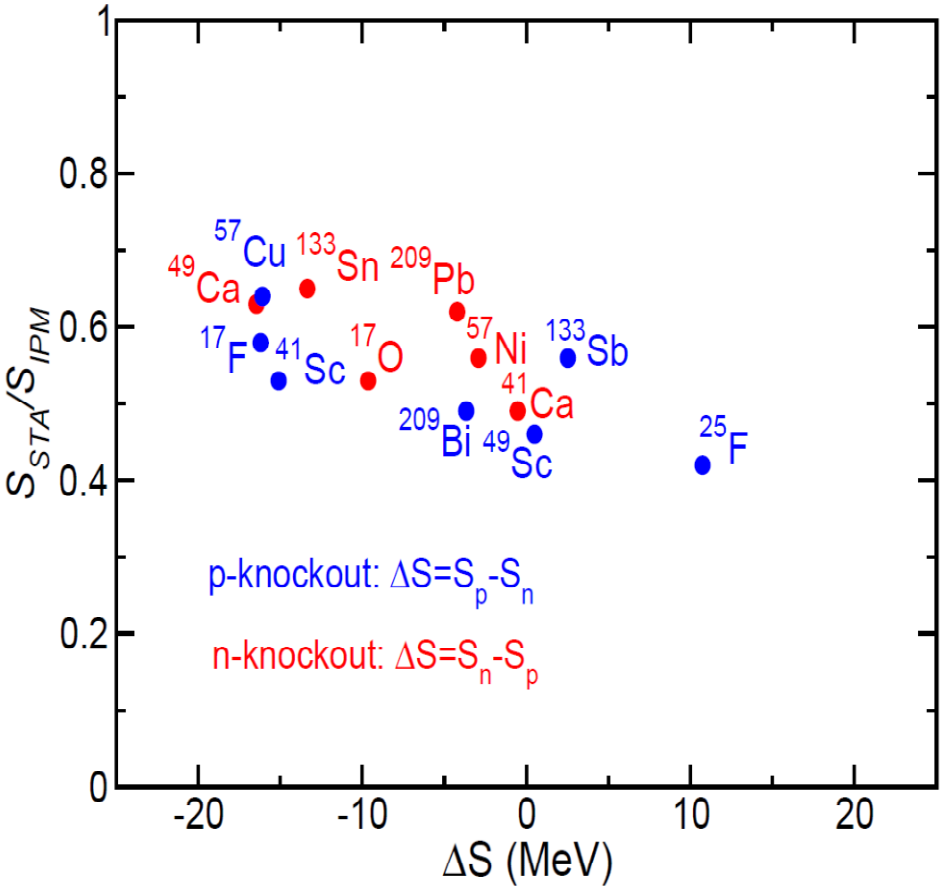
N.K. Timofeyuk, Phys. Rev. C ***84****, 054313 (2011)*

Spectroscopic factor reduction from IPM values

Removing one nucleon



Adding one nucleon



Conclusions:

Overlap integrals, SFs and ANCs must be calculated from solution of inhomogeneous equation with a properly chosen source term. Advantages are:

- exact asymptotic behaviour is guaranteed
- information about normalization is not lost
- It allows small model spaces to be used to explain large reduction of spectroscopic strength due to the coupling to missing model spaces.

STA can reconcile reduction of spectroscopic strength in double closed shell nuclei with double magic nature of these nuclei.

Implications for the meaning of spectroscopic factors:

SFs are the measure of strength of the interaction of the removed nucleon rather than the measure of the shell occupancies.

Publications:

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