Wilson loops and amplitudes in N=4 Super Yang-Mills

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Frascati 7 March 2011

N=4 Super Yang-Mills

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  conformally invariant, β fn. = 0
  - spin 1 gluon
    4 spin 1/2 gluinos
    6 spin 0 real scalars

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  - only planar diagrams
- AdS/CFT duality — Maldacena 97
  - large-λ limit of 4dim CFT $\leftrightarrow$ weakly-coupled string theory

    (aka weak-strong duality)

AdS/CFT duality, amplitudes & Wilson loops

- planar scattering amplitude at strong coupling   Alday Maldacena 07

$$\mathcal{M} \sim \exp\left[i \frac{\sqrt{\lambda}}{2\pi} (Area)_{cl} \right]$$

area of string world-sheet   (classical solution
neglect $O(1/\sqrt{\lambda})$ corrections)

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$$M_n = M_n^{(0)} \exp\left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon)\, m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right]$$

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- computation ``formally the same as ... the expectation value of a Wilson loop given by a sequence of light-like segments”

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Brandhuber Heslop Travaglini 07

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- no amplitudes are known beyond the 6-point 2-loop amplitude

MHV amplitudes in planar N=4 SYM

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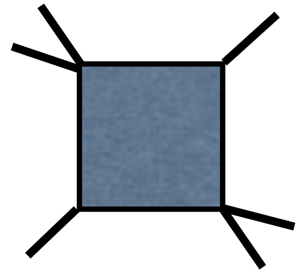
$$m_n^{(1)} = \sum_{pq} F^{2\mathrm{me}}(p, q, P, Q) \qquad n \geq 6$$



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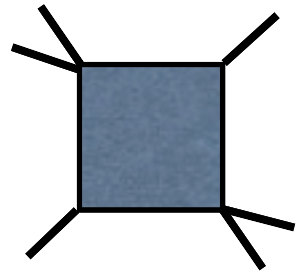

- at 2 loops, iteration formula for the *n*-pt amplitude

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + R$$

Anastasiou Bern Dixon Kosower 03

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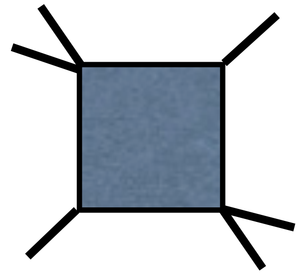

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- at all loops, ansatz for a resummed exponent

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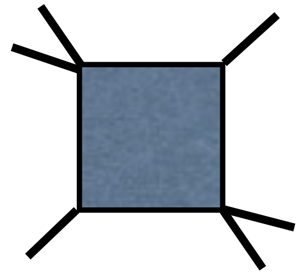

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remainder function

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ansatz for MHV amplitudes in planar N=4 SYM

Bern Dixon Smirnov 05

$$M_n = M_n^{(0)} \left[1 + \sum_{L=1}^{\infty} a^L m_n^{(L)}(\epsilon) \right]$$

$$= M_n^{(0)} \exp \left[\sum_{l=1}^{\infty} a^l \left(f^{(l)}(\epsilon) m_n^{(1)}(l\epsilon) + Const^{(l)} + E_n^{(l)}(\epsilon) \right) \right]$$

coupling $a = \frac{\lambda}{8\pi^2} (4\pi e^{-\gamma})^{\epsilon}$ $\lambda = g^2 N$ 't Hooft parameter

$$f^{(l)}(\epsilon) = \frac{\hat{\gamma}_K^{(l)}}{4} + \epsilon \, \frac{l}{2} \hat{G}^{(l)} + \epsilon^2 \, f_2^{(l)} \qquad E_n^{(l)}(\epsilon) = O(\epsilon)$$

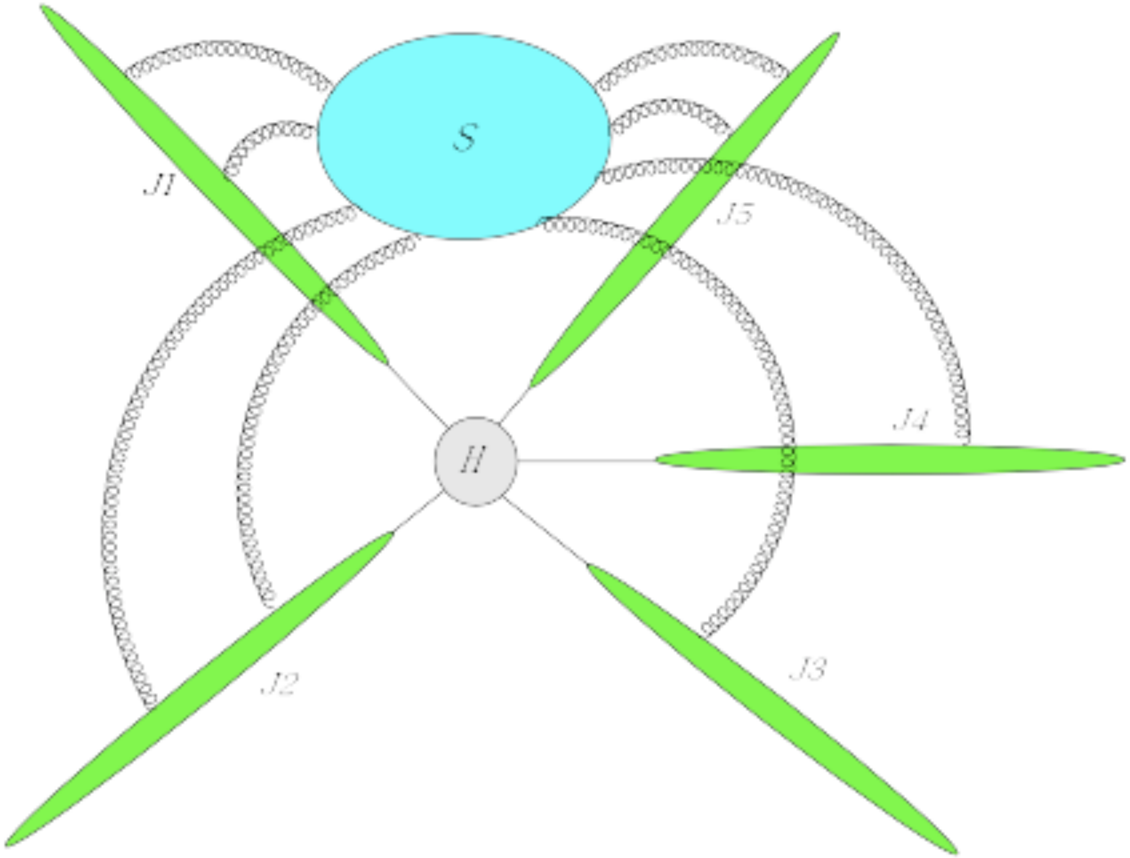
$\hat{\gamma}_K^{(l)}$ cusp anomalous dimension, known to all orders of *a* — Korchemsky Radyuskin 86; Beisert Eden Staudacher 06

$\hat{G}^{(l)}$ collinear anomalous dimension, known through O(a^4) — Bern Dixon Smirnov 05; Cachazo Spradlin Volovich 07

ansatz generalises the iteration formula for the 2-loop *n*-pt amplitude $m_n^{(2)}$

$$m_n^{(2)}(\epsilon) = \frac{1}{2} \left[m_n^{(1)}(\epsilon) \right]^2 + f^{(2)}(\epsilon) \, m_n^{(1)}(2\epsilon) + Const^{(2)} + \mathcal{O}(\epsilon)$$

Factorisation of a multi-leg amplitude in QCD



Mueller 1981
Sen 1983
Botts Sterman 1987
Kidonakis Oderda Sterman 1998
Catani 1998
Tejeda-Yeomans Sterman 2002
Kosower 2003
Aybat Dixon Sterman 2006
Becher Neubert 2009
Gardi Magnea 2009

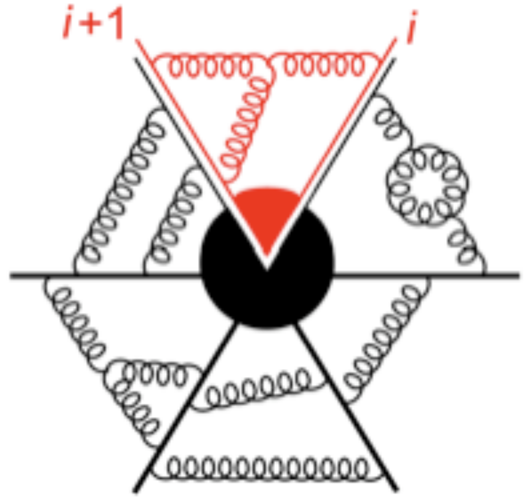
$$\mathcal{M}_N(p_i/\mu,\epsilon) = \sum_L \mathcal{S}_{NL}(\beta_i\cdot\beta_j,\epsilon)\, H_L\left(\frac{2p_i\cdot p_j}{\mu^2}, \frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2}\right) \prod_i \frac{J_i\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i\cdot n_i)^2}{n_i^2},\epsilon\right)}$$

$p_i = \beta_i Q_0/\sqrt{2}$ value of Q_0 is immaterial in *S*, *J*

to avoid double counting of soft-collinear region (IR double poles),
$\mathcal{J}_i$ removes eikonal part from J_i, which is already in S
$J_i/\mathcal{J}_i$ contains only single collinear poles

$\mathcal{N}=4$ SYM in the planar limit

- colour-wise, the planar limit is trivial: can absorb S into J_i



- each slice is square root of Sudakov form factor

$$\mathcal{M}_n = \prod_{i=1}^{n} \left[\mathcal{M}^{[gg\to 1]} \left(\frac{s_{i,i+1}}{\mu^2}, \alpha_s, \epsilon \right) \right]^{1/2} h_n(\{p_i\}, \mu^2, \alpha_s, \epsilon)$$

- β fn = 0 ⇒ coupling runs only through dimension $\bar{\alpha}_s(\mu^2)\mu^{2\epsilon} = \bar{\alpha}_s(\lambda^2)\lambda^{2\epsilon}$

Sudakov form factor has simple solution

$$\ln \left[\Gamma \left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] = -\frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{\alpha_s(\mu^2)}{\pi} \right)^n \left(\frac{-Q^2}{\mu^2} \right)^{-n\epsilon} \left[\frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right]$$

⇒ IR structure of $\mathcal{N}=4$ SUSY amplitudes

Magnea Sterman 90
Bern Dixon Smirnov 05

Brief history of the ansatz

the ansatz checked for the 3-loop 4-pt amplitude — Bern Dixon Smirnov 05

2-loop 5-pt amplitude — Cachazo Spradlin Volovich 06; Bern Czakon Kosower Roiban Smirnov 06

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at 2 loops, the remainder function characterises the deviation from the ansatz

$$R_n^{(2)} = m_n^{(2)}(\epsilon) - \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 - f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) - Const^{(2)}$$

$R_6^{(2)}$ known numerically — Bern Dixon Kosower Roiban Spradlin Vergu Volovich 08; Drummond Henn Korchemsky Sokatchev 08; Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

analitically — Duhr Smirnov VDD 09

Wilson loops & Ward identities

Drummond Henn Korchemsky Sokatchev 07

- *N*=4 SYM is invariant under *SO*(2,4) conformal transformations

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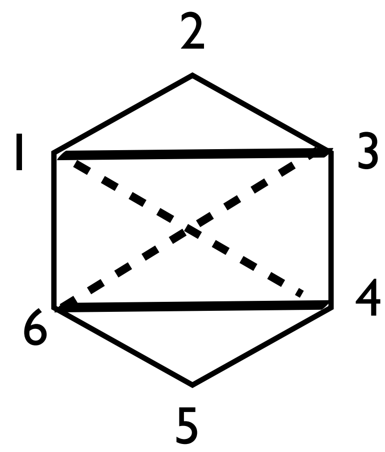
- for n = 6, the conformally invariant cross ratios are

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2} \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}$$



x_i are variables in a dual space s.t. $p_i = x_i - x_{i+1}$

thus $x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$

Wilson loops

- $$W[\mathcal{C}_n] = \mathrm{Tr}\; \mathcal{P} \exp \left[ig \oint \mathrm{d}\tau \dot{x}^\mu(\tau) A_\mu(x(\tau)) \right]$$

closed contour $\mathcal{C}_n$ made by light-like external momenta

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  Alday Maldacena 07

- non-Abelian exponentiation theorem: vev of Wilson loop as an exponential, allows us to compute the log of W

  Gatheral 83
  Frenkel Taylor 84

  $$\langle W[\mathcal{C}_n] \rangle = 1 + \sum_{L=1}^{\infty} a^L W_n^{(L)} = \exp \sum_{L=1}^{\infty} a^L w_n^{(L)}$$

  through 2 loops $\quad w_n^{(1)} = W_n^{(1)} \qquad w_n^{(2)} = W_n^{(2)} - \frac{1}{2} \left(W_n^{(1)} \right)^2$

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- relation between 1 loop amplitudes & Wilson loops

  $$w_n^{(1)} = \frac{\Gamma(1-2\epsilon)}{\Gamma^2(1-\epsilon)} m_n^{(1)} = m_n^{(1)} - n\frac{\zeta_2}{2} + \mathcal{O}(\epsilon)$$

  Brandhuber Heslop Travaglini 07

Wilson loops

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  - at 2 loops

$$w_n^{(2)}(\epsilon) = f_{WL}^{(2)}(\epsilon)\, w_n^{(1)}(2\epsilon) + C_{WL}^{(2)} + R_{n,WL}^{(2)} + \mathcal{O}(\epsilon)$$

with $f_{WL}^{(2)}(\epsilon) = -\zeta_2 + 7\zeta_3\epsilon - 5\zeta_4\epsilon^2$

(to be compared with $f^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\epsilon - \zeta_4\epsilon^2$ for the amplitudes)

$$R_{4,WL} = R_{5,WL} = 0$$

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- $R_{n,WL}^{(2)}$ arbitrary function of conformally invariant cross ratios

$$u_{ij} = \frac{x_{ij+1}^2 x_{i+1j}^2}{x_{ij}^2 x_{i+1j+1}^2} \qquad \text{with} \qquad x_{k,k+r}^2 = (p_k + \ldots + p_{k+r-1})^2$$

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- duality Wilson loop ⇔ MHV amplitude is expressed by

  $$R_{n,WL}^{(2)} = R_n^{(2)}$$

Brief history of 2-loop Wilson loops

4-edged Wilson loop	Drummond Henn Korchemsky Sokatchev 07
5-edged Wilson loop	Drummond Henn Korchemsky Sokatchev 07
6-edged Wilson loop (numeric)	Drummond Henn Korchemsky Sokatchev 08 Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09
6-edged Wilson loop (analytic)	Duhr Smirnov VDD 09
n-edged Wilson loop (numeric)	Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

- checked that $R_n = R_n(u_{ij})$
  checked multi-collinear limits

Collinear limits of Wilson loops

collinear limit $a||b$

Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

$R_6 \to 0$ $\quad$ $R_7 \to R_6$ $\quad$ $R_n \to R_{n-1}$

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triple collinear limit $a||b||c$

$R_6 \rightarrow R_6$ $\quad R_7 \rightarrow R_6$ $\quad R_8 \rightarrow R_6 + R_6$ $\quad R_n \rightarrow R_{n-2} + R_6$

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quadruple collinear limit $a||b||c||d$

$R_7 \to R_7$ $\quad R_8 \to R_7$ $\quad R_9 \to R_6 + R_7$ $\quad R_n \to R_{n-3} + R_7$

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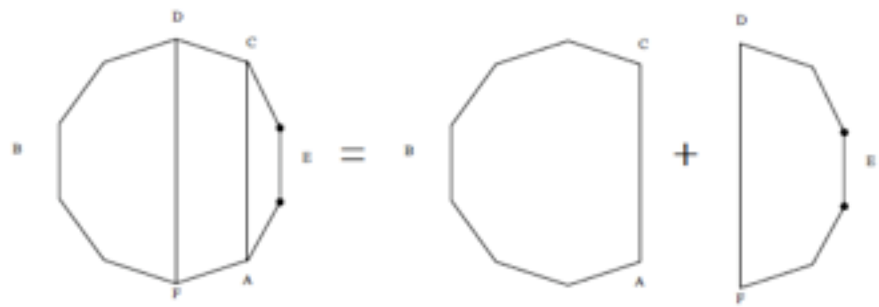
(k+1)-ple collinear limit $i_1||i_2||\cdots||i_{k+1}$

$R_n \to R_{n-k} + R_{k+4}$



(n-4)-ple collinear limit $i_1||i_2||\cdots||i_{n-4}$

$R_{n-1} \to R_{n-1}$ $\quad$ $R_n \to R_{n-1}$

(n-3)-ple collinear limit $i_1||i_2||\cdots||i_{n-3}$

$R_n \to R_n$

Collinear limits of Wilson loops

collinear limit $a||b$ Anastasiou Brandhuber Heslop Khoze Spence Travaglini 09

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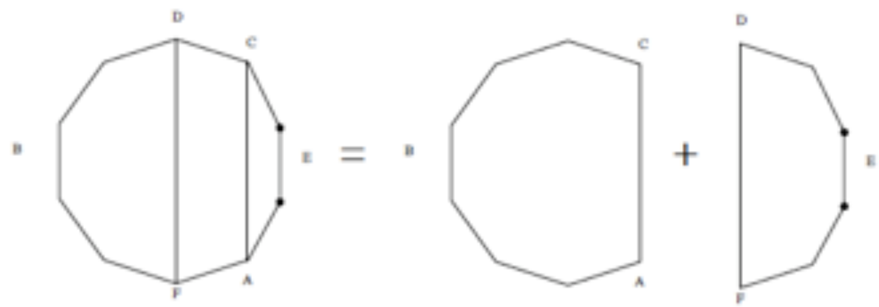


(n-4)-ple collinear limit $i_1||i_2||\cdots||i_{n-4}$

$R_{n-1} \to R_{n-1}$ $R_n \to R_{n-1}$

(n-3)-ple collinear limit $i_1||i_2||\cdots||i_{n-3}$

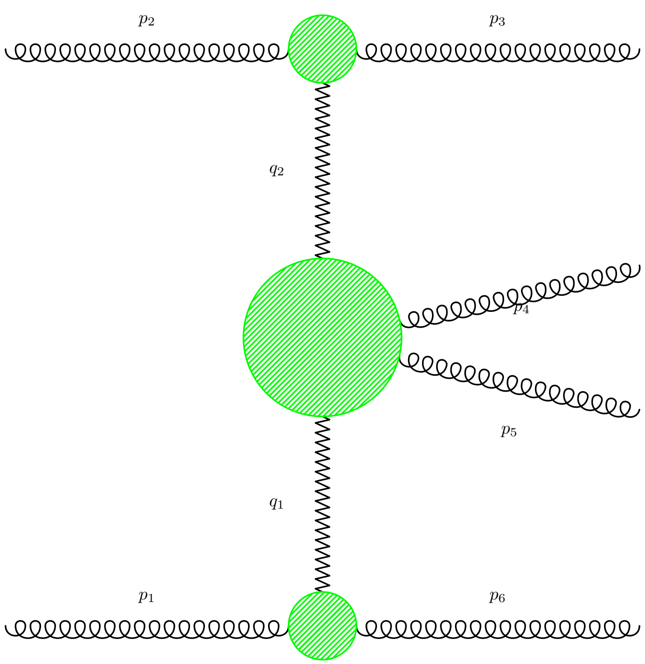
$R_n \to R_n$

- thus R_n is fixed by the (n-3)-ple collinear limit

Quasi-multi-Regge limit of hexagon Wilson loop

- 6-pt amplitude in the qmR limit of a pair along the ladder

$$y_3 \gg y_4 \simeq y_5 \gg y_6; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}|$$



the conformally invariant cross ratios are

$$u_{36} = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12} s_{45}}{s_{123} s_{345}}$$

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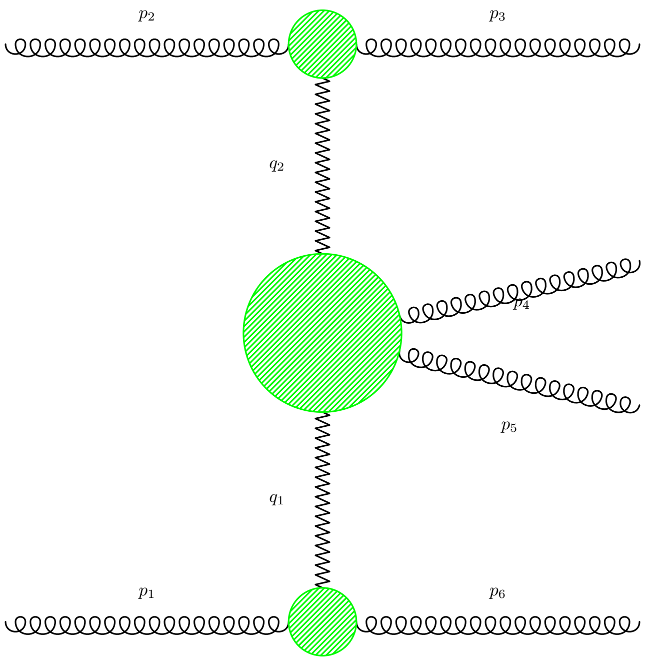

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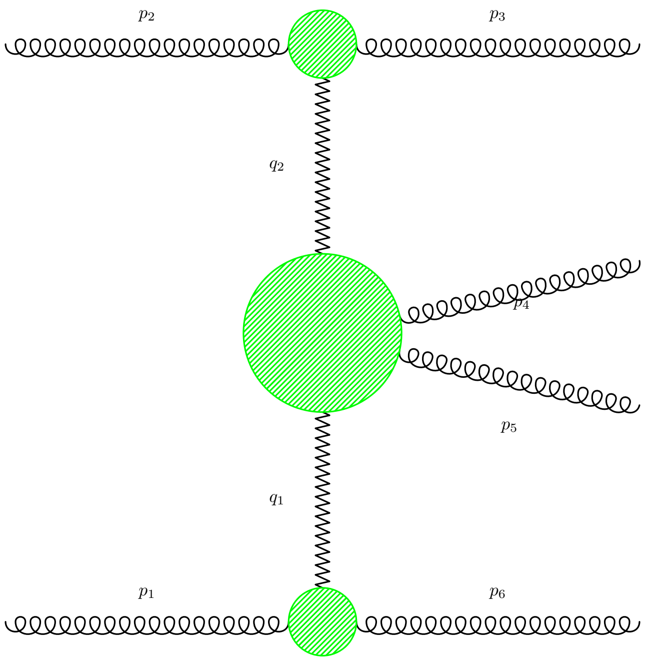

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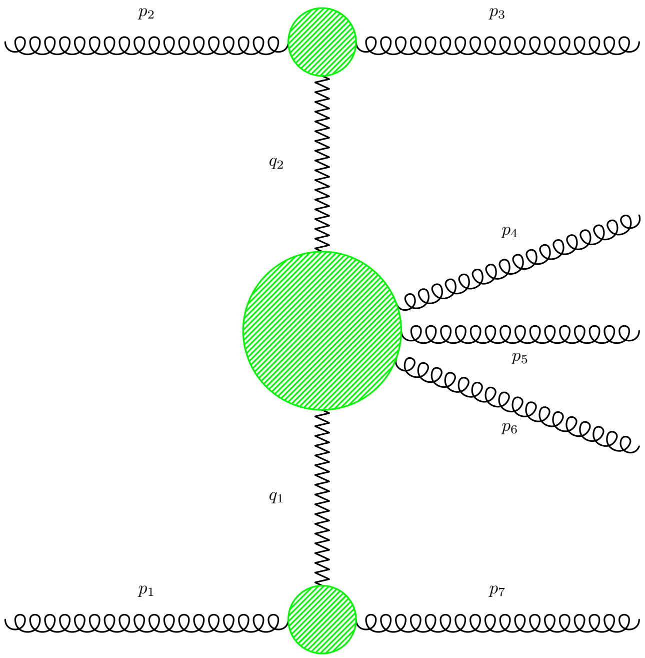
Duhr Glover Smirnov VDD 08

Quasi-multi-Regge limit of n-sided Wilson loop

- 7-pt amplitude in the qmR limit of a triple along the ladder

$$y_3 \gg y_4 \simeq y_5 \simeq y_6 \gg y_7; \qquad |p_{3\perp}| \simeq |p_{4\perp}| \simeq |p_{5\perp}| \simeq |p_{6\perp}| \simeq |p_{7\perp}|$$



7 cross ratios, which are all $O(1)$
R_7 is invariant under the qmR limit of a triple along the ladder

Quasi-multi-Regge limit of n-sided Wilson loop

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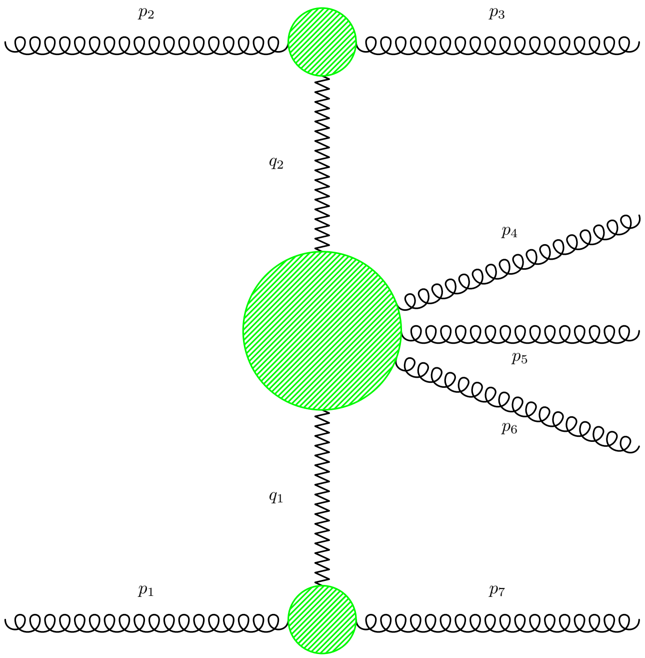

7 cross ratios, which are all $O(1)$
R_7 is invariant under the qmR limit of a triple along the ladder

- can be generalised to the n-pt amplitude in the qmR limit of a (n-4)-ple along the ladder

$$y_3 \gg y_4 \simeq \ldots \simeq y_{n-1} \gg y_n; \qquad |p_{3\perp}| \simeq \ldots \simeq |p_{n\perp}|$$

Duhr Smirnov VDD 09

Quasi-multi-Regge limit of Wilson loops

- *L*-loop Wilson loops are Regge exact

Drummond Korchemsky Sokatchev 07
Duhr Smirnov VDD 09

$$w_n^{(L)}(\epsilon) = f_{WL}^{(L)}(\epsilon)\, w_n^{(1)}(L\epsilon) + C_{WL}^{(L)} + R_{n,WL}^{(L)}(u_{ij}) + \mathcal{O}(\epsilon)$$

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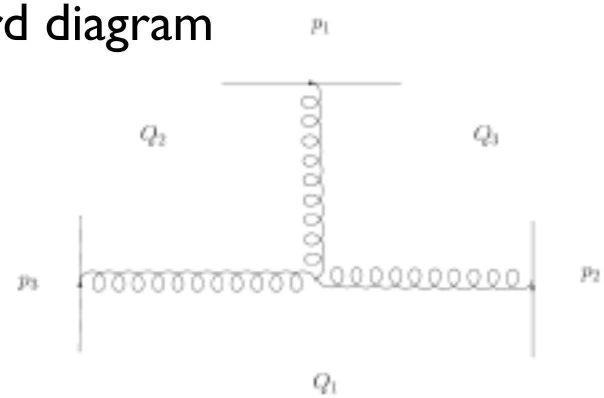
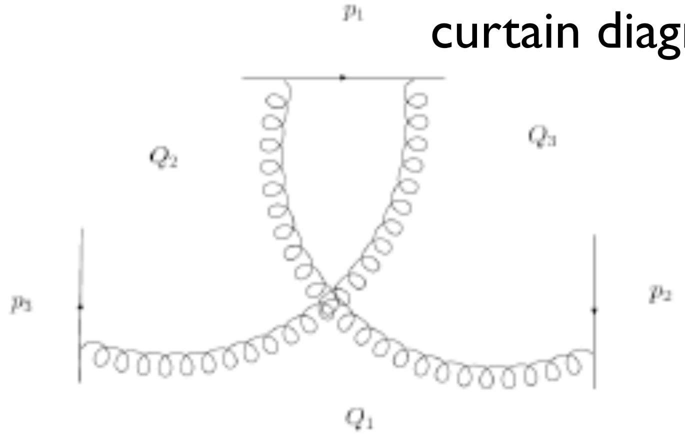
- we may compute the Wilson loop in qmRk
  the result will be correct in general kinematics !!!

Diagrams of 2-loop Wilson loops

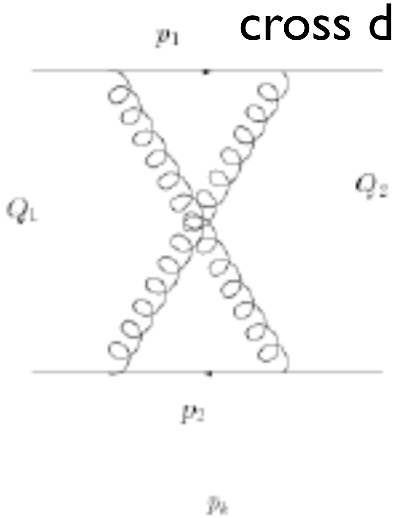
hard diagram



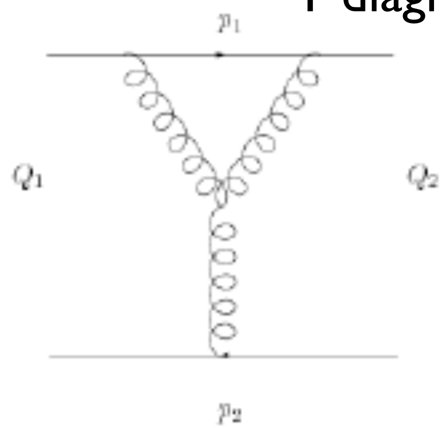
curtain diagram

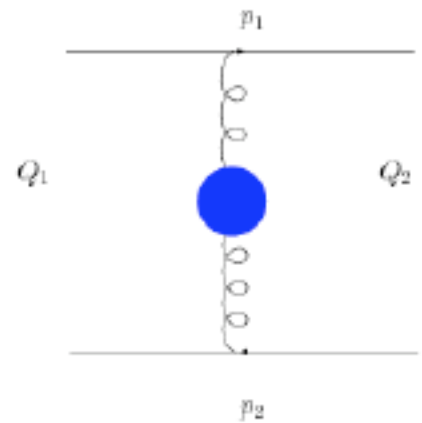


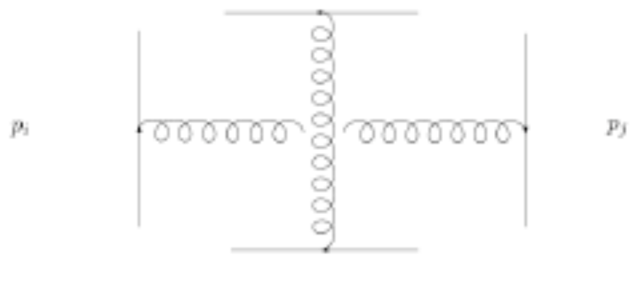
cross diagram



Y diagram







factorised cross diagram

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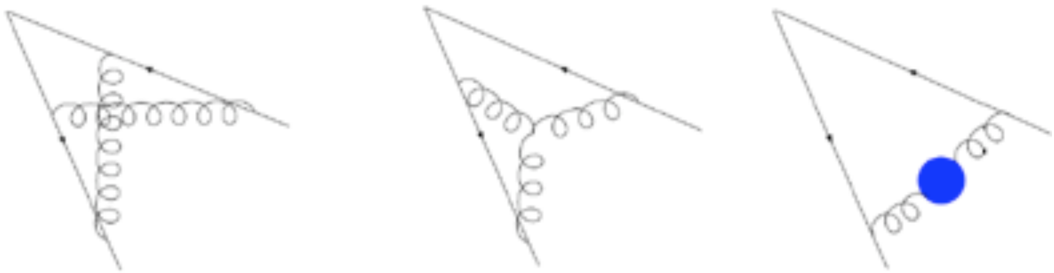
each diagram yields an integral,
similar to a Feynman-parameter integral

Computing 2-loop Wilson loops

cusp diagrams are given by cross and Y diagrams with gluons attaching to consecutive sides



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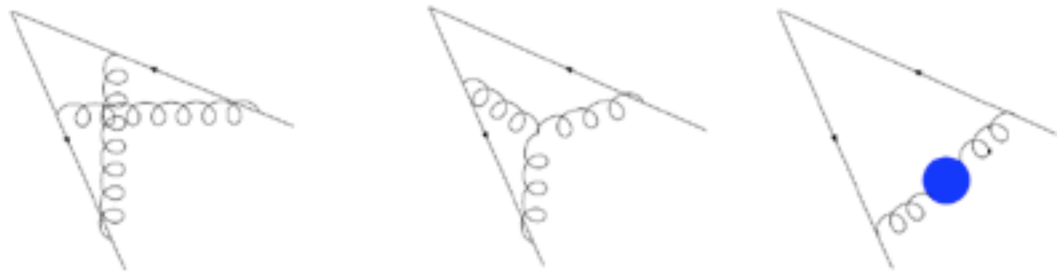
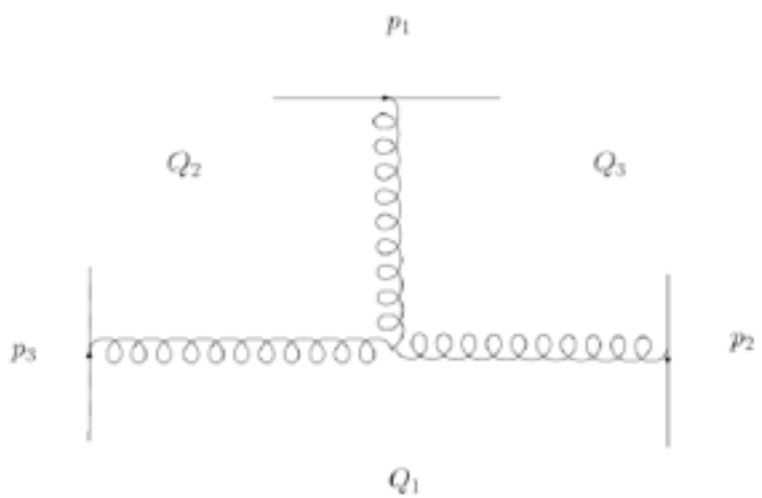


- most difficult diagrams to compute are hard diagrams



f_H has $1/\varepsilon^2$ singularities if $Q_1 = Q_2 = 0$, $Q_3 \neq 0$
it has $1/\varepsilon$ singularities if $Q_1 = 0$, $Q_2, Q_3 \neq 0$
it is finite if $Q_1, Q_2, Q_3 \neq 0$

e.g. for $n=6$, *the* most difficult diagram is

$f_H(p_1, p_3, p_5; p_4, p_6, p_2)$ which is finite

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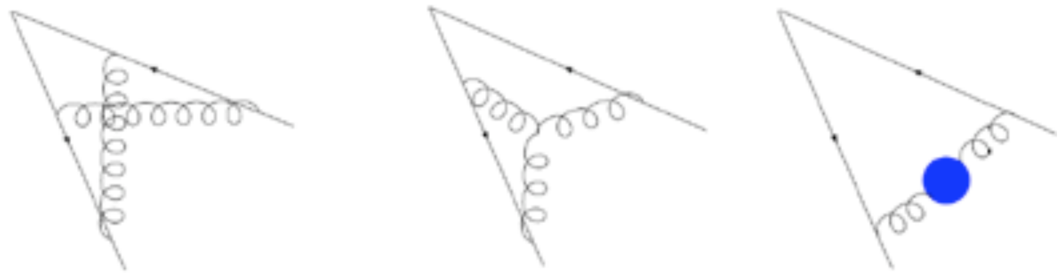

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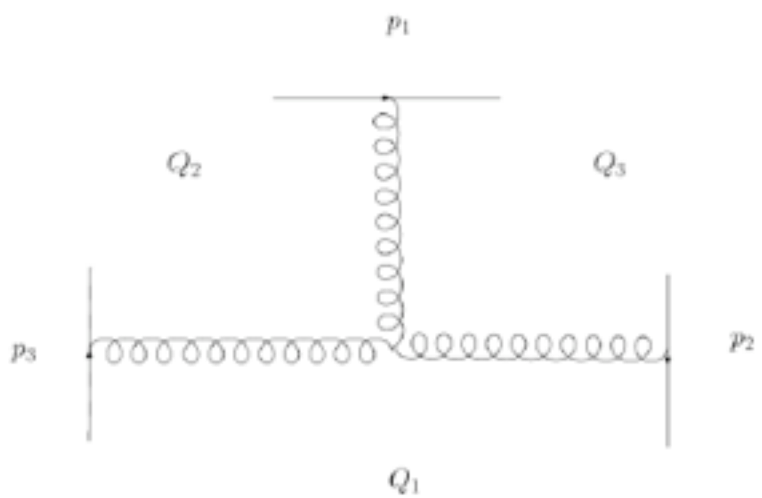

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- most general hard diagram has $Q_1^2, Q_2^2, Q_3^2 \neq 0$; it occurs for $n \geq 9$

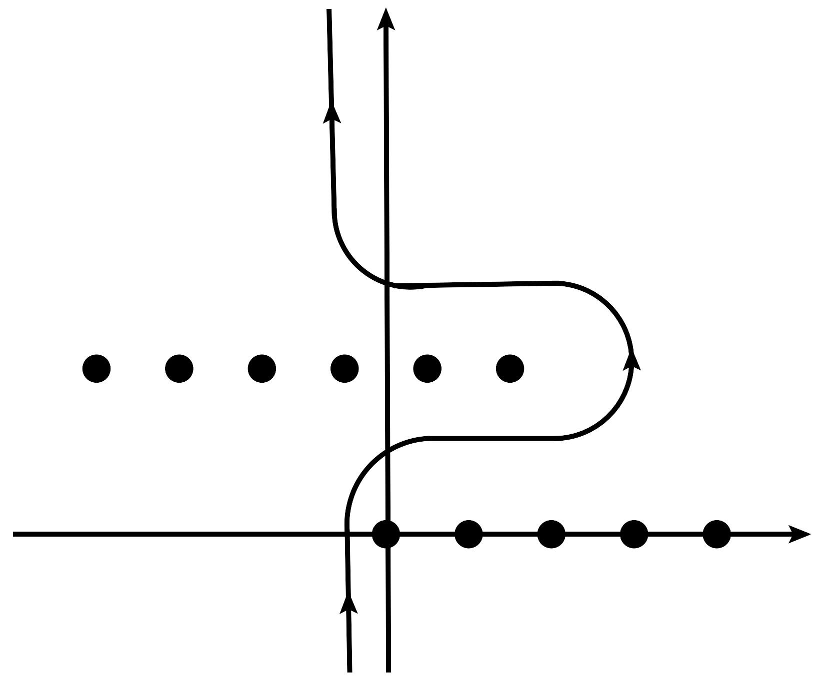
Wilson loops: analytic calc

1. Use Mellin-Barnes (MB) representation of the Feynman-parameter integrals: replace each denominator by a contour integral

$$\frac{1}{(A+B)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathrm{d}z\, \Gamma(-z)\, \Gamma(\lambda+z)\, \frac{A^z}{B^{\lambda+z}}$$

integral turns into a sum of residues

$$\mathrm{Res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$$



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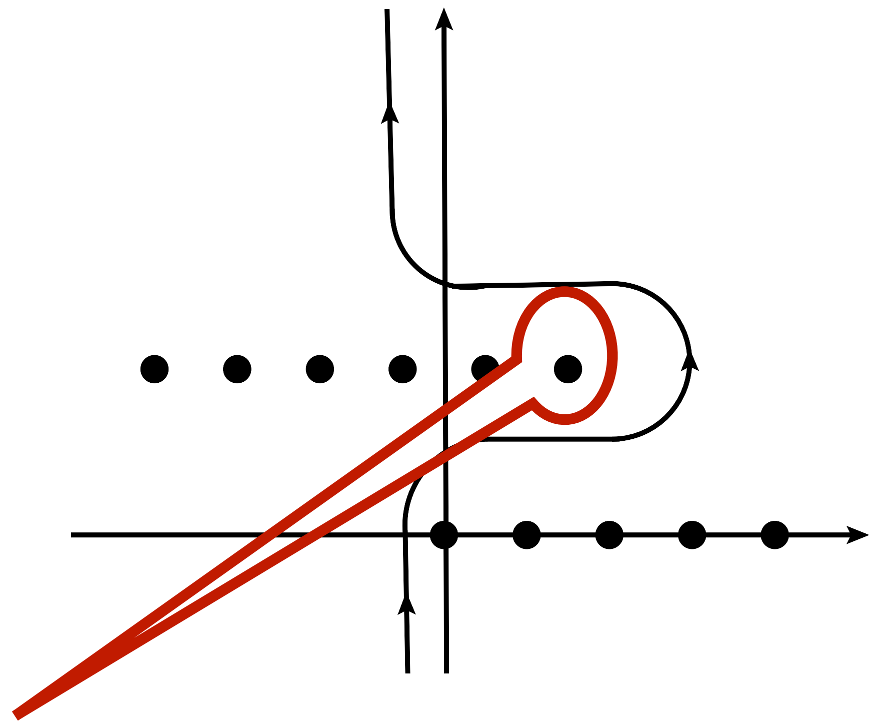
2. Use Regge exactness in the qmR limit: retain only leading behaviour (*i.e.* leading residues) of the integral



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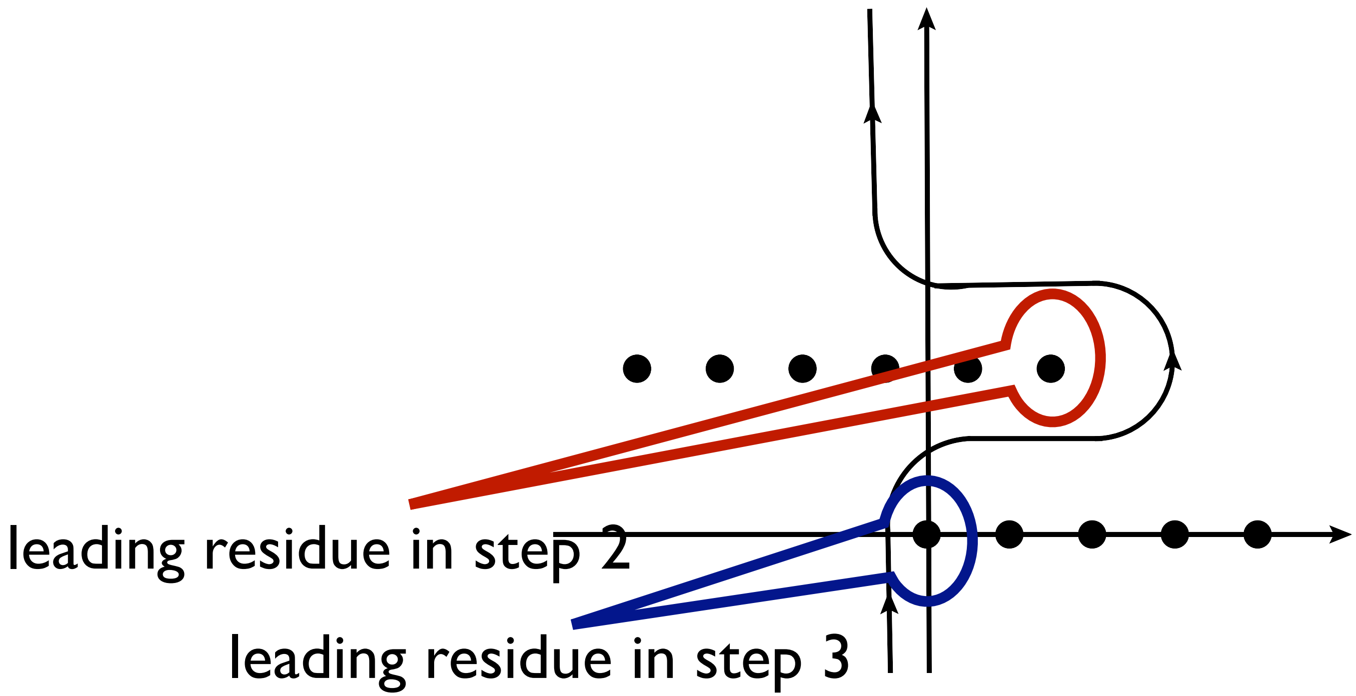

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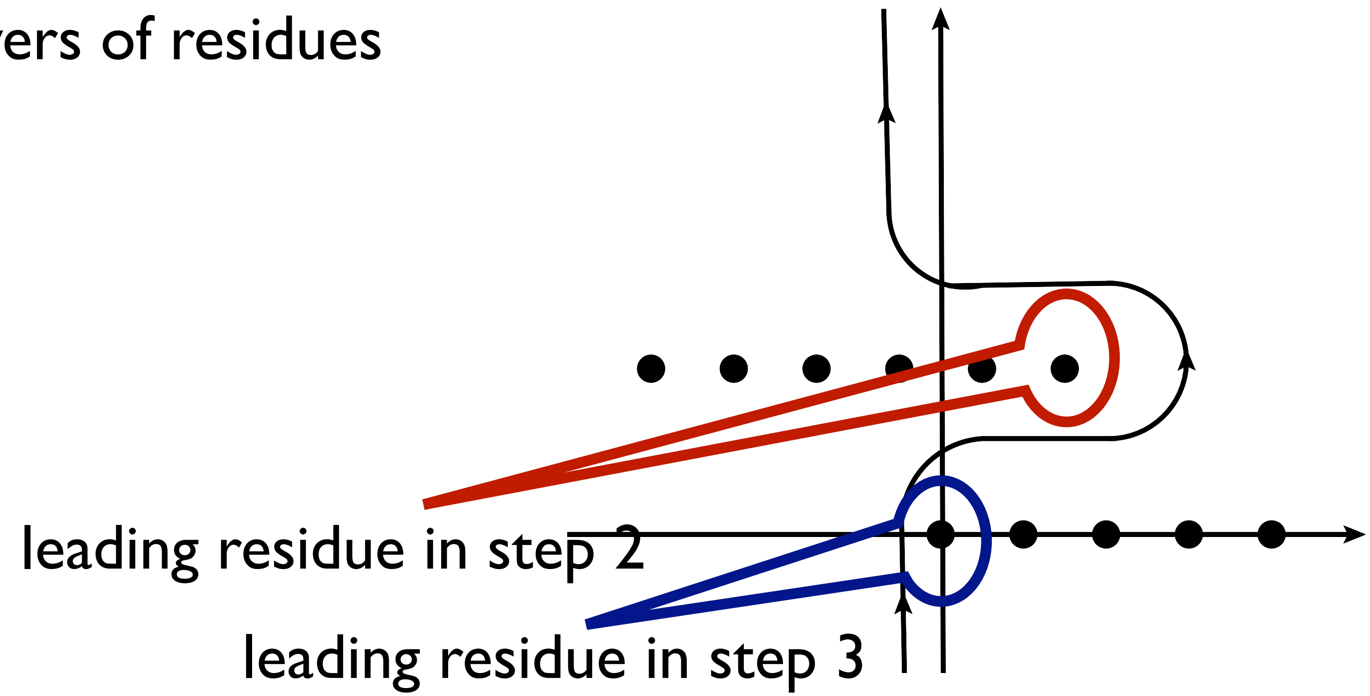

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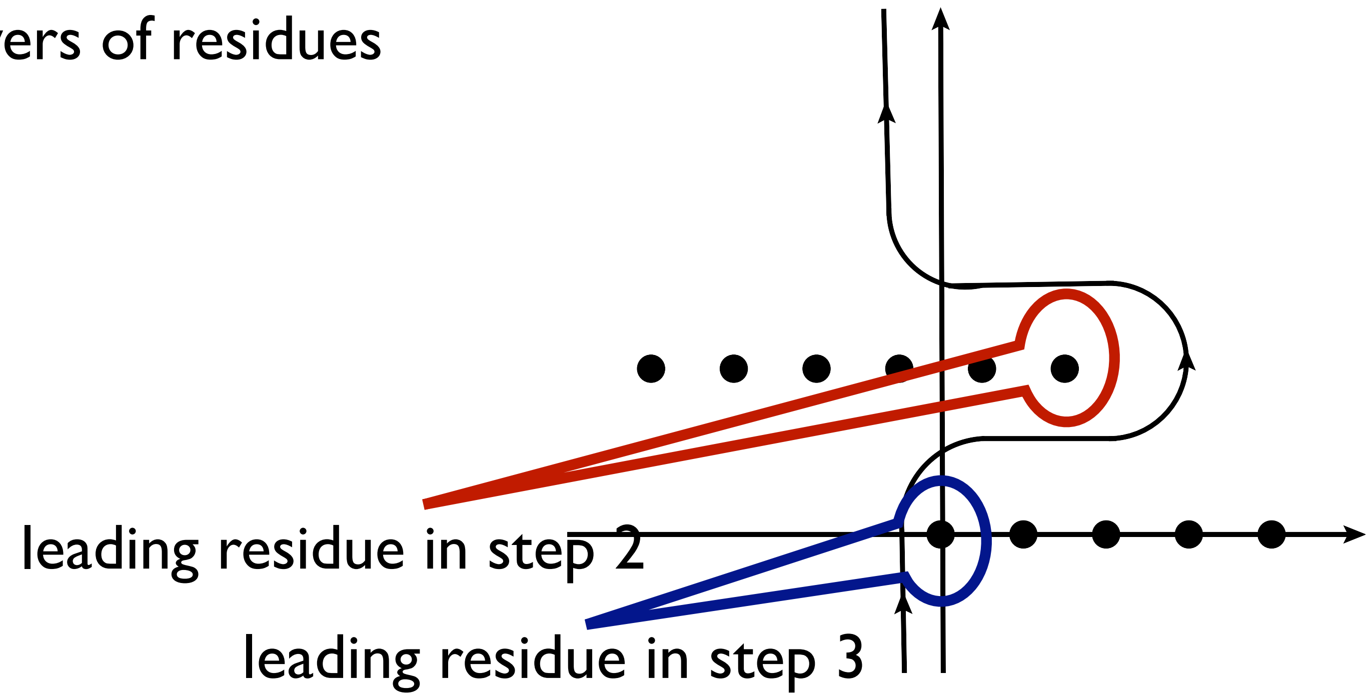

in general, get nested harmonic sums → Goncharov polylogarithms

$$\sum_{n_1=1}^{\infty} \frac{u_1^{n_1}}{n_1^{m_1}} \sum_{n_2=1}^{n_1-1} \cdots \sum_{n_k=1}^{n_{k-1}-1} \frac{u_k^{n_k}}{n_k^{m_k}} = (-1)^k\, G\left(\underbrace{0,\ldots,0}_{m_1-1}, \frac{1}{u_1}, \ldots, \underbrace{0,\ldots,0}_{m_k-1}, \frac{1}{u_1\ldots u_k}; 1\right)$$

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$$\int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}z_1}{2\pi i} \frac{\mathrm{d}z_2}{2\pi i} \frac{\mathrm{d}z_3}{2\pi i} (z_1\, z_2 + z_2\, z_3 + z_3\, z_1)\, u_1^{z_1}\, u_2^{z_2}\, u_3^{z_3}$$
$$\times \Gamma(-z_1)^2\, \Gamma(-z_2)^2\, \Gamma(-z_3)^2\, \Gamma(z_1 + z_2)\, \Gamma(z_2 + z_3)\, \Gamma(z_3 + z_1)$$

the result is in terms of Goncharov polylogarithms

$$G(a, \vec{w}; z) = \int_0^z \frac{\mathrm{d}t}{t-a}\, G(\vec{w}; t)\,, \qquad G(a; z) = \ln\left(1 - \frac{z}{a}\right)$$

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- the remainder function $R_6^{(2)}$ is given in terms of $O(10^3)$ Goncharov polylogarithms $G(u_1, u_2, u_3)$

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2-loop 6-edged remainder function $R_6^{(2)}$

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↓ finite answer, but in intermediate steps many divergences
output is punishingly long

our result has been simplified and given in terms of polylogarithms

Goncharov Spradlin Vergu Volovich 10

$$
\begin{aligned}
R^{(2)}_{6,WL}(u_1,u_2,u_3) &= \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) \\
&- \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}
\end{aligned}
$$

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Goncharov Spradlin Vergu Volovich 10

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{J^4}{24} + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

where

$$x_i^{\pm} = u_i x^{\pm} \qquad x^{\pm} = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3} \qquad \Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$

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↑ answer is short and simple
introduces the *theory of motives* in TH physics

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the symbol is $\mathrm{Sym}[T_k] = R_1 \otimes \cdots \otimes R_k$

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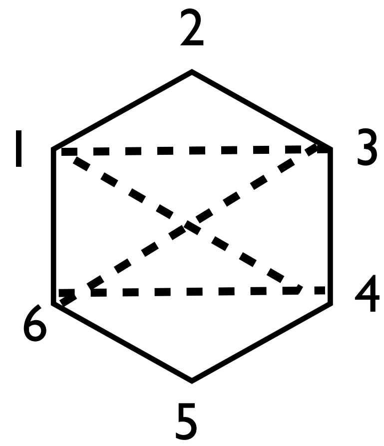
$\Longrightarrow$ a symbol determines a polynomial of uniform degree up to a constant

Z_n symmetric regular hexagons

regular hexagons are characterised by



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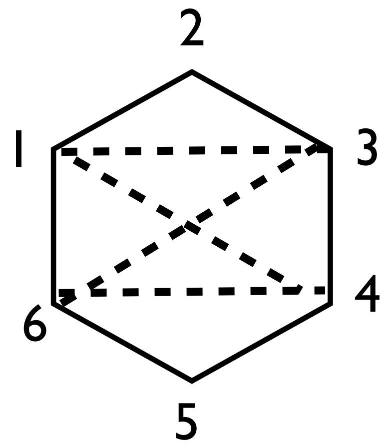

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At strong coupling, remainder function is obtained from ``minimal area surfaces in $AdS_5$ which end on a null polygonal contour at the boundary". One gets ``integral equations which determine the area as a function of the shape of the polygon. The equations are identical to those of the Thermodynamics Bethe Ansatz. The area is given by the free energy of the TBA system. The high temperature limit of the TBA system can be exactly solved"

$$R_6^{strong}(u,u,u) = \underbrace{\frac{\pi}{6} - \frac{1}{3\pi}\phi^2}_{\text{free energy}} - \frac{3}{8}\underbrace{\left(\ln^2(u) + 2\,\mathrm{Li}^2(1-u)\right)}_{\text{BDS - BDSlike}} \qquad u = \frac{1}{4\cos^2(\phi/3)}$$

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Remainder function at weak and strong coupling

compare remainder functions at weak and strong coupling introducing coefficients in the strong coupling result and try to curve fit the 2 results

$$R_6^{strong}(u,u,u) = c_1\left(\frac{\pi}{6} - \frac{1}{3\pi}\phi^2\right) + c_2\left(\frac{3}{8}\left(\ln^2(u) + 2\,\mathrm{Li}^2(1-u)\right)\right) + c_3$$

$$c_1 = 0.263\pi^3 \qquad c_2 = 0.860\pi^2 \qquad c_3 = -\frac{\pi^2}{12}c_2$$

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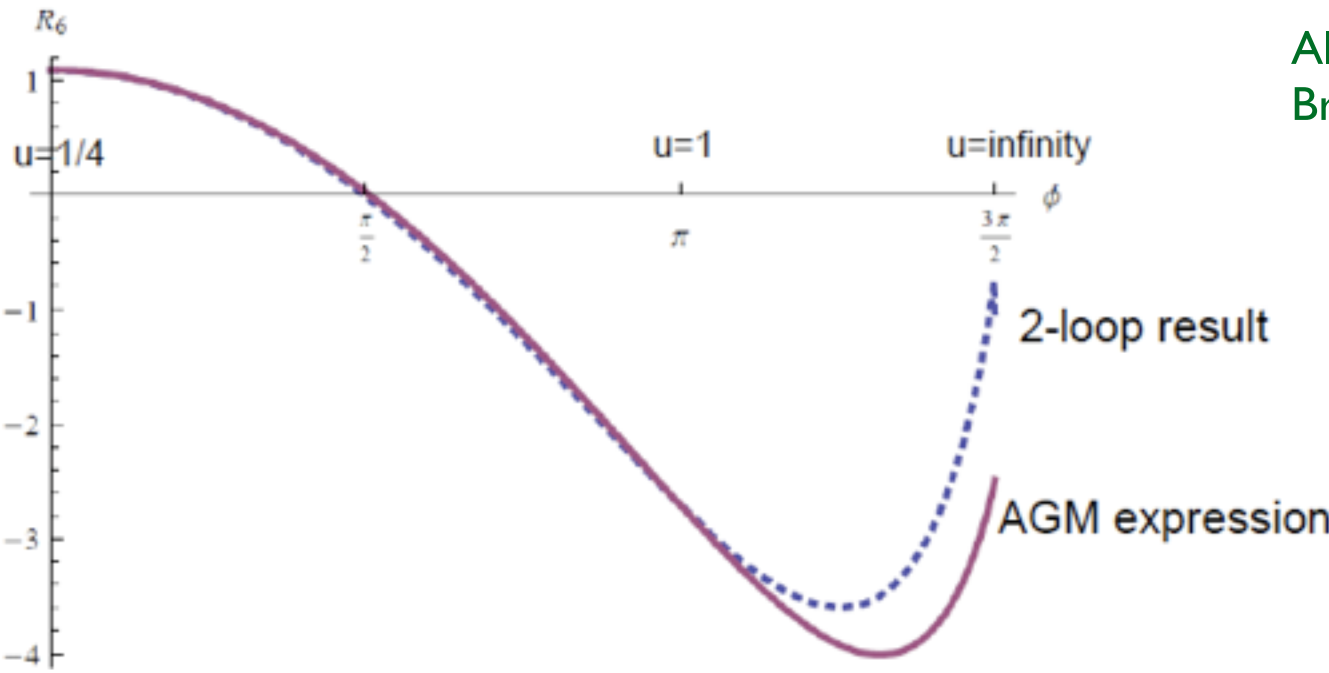

- the 2 curves are strikingly similar

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$$R_{6,WL}^{(2)}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) = -\tfrac{105}{64}\zeta_3\log 2 - \tfrac{5}{64}\log^4 2 + \tfrac{5}{64}\pi^2\log^2 2 - \tfrac{15}{8}\mathrm{Li}_4\left(\tfrac{1}{2}\right) + \tfrac{17\pi^4}{2304}$$

uniform, and intrinsic, weight 4

One-loop amplitude squared

the 2-loop *n*-pt amplitude is

$$m_n^{(2)}(\epsilon) = \frac{1}{2}\left[m_n^{(1)}(\epsilon)\right]^2 + f^{(2)}(\epsilon)\, m_n^{(1)}(2\epsilon) + Const^{(2)} + R$$

what about that ?

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Alday Henn Plefka Schuster 09

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↓ not practical for phenomenology (where DR rules the waves)

Amplitudes in twistor space

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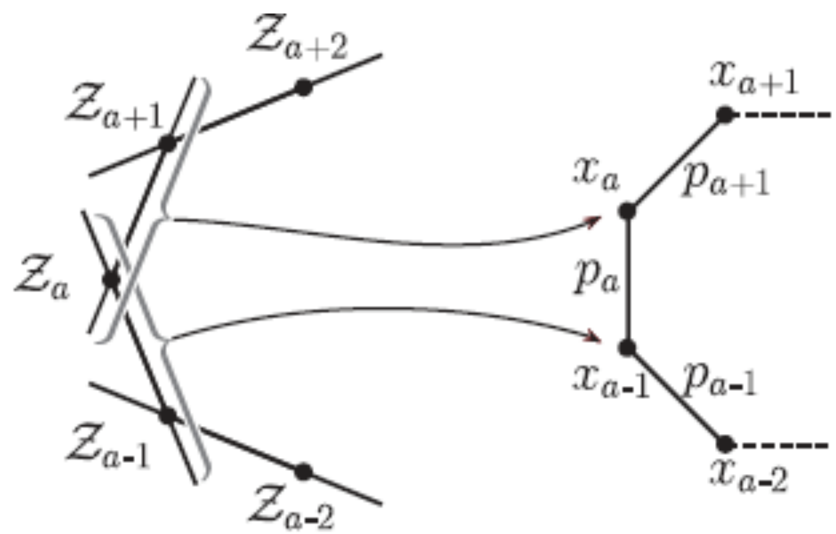

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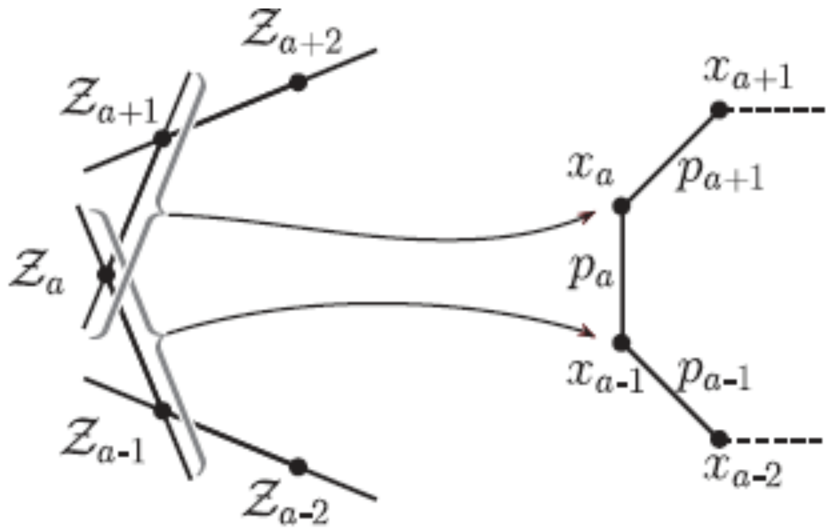

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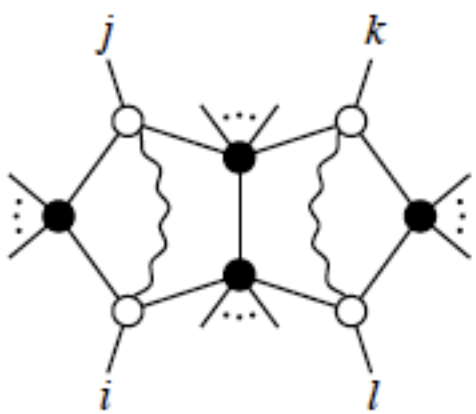
2-loop *n*-pt MHV amplitudes can be written as sum of pentaboxes in twistor space

$$m_n^{(2)} = \frac{1}{2} \sum_{i<j<k<l<i}$$



Arkani-Hamed Bourjaily Cachazo Trnka10

8-edged Wilson loop in AdS_3

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where $\chi^+ = e^{2\pi \mathrm{Im}\, m}$ $\quad \chi^- = e^{-2\pi \mathrm{Re}\, m}$ $\quad m = |m|e^{i\phi}$

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Alday Maldacena 09

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- 2-loop 2*n*-sided polygon *R* conjectured through collinear limits Heslop Khoze 10
  proven through OPE Gaiotto Maldacena Sever Vieira 10

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- analytic comparison between the 2-loop 6-edged Wilson loop at weak and strong couplings
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- more is to come ... stay tuned!