Dissipation and Quantization

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- 't Hooft proposal for Deterministic Quantum Mechanics (with information loss)*;
- System of two dissipative oscillators†;
- Electromagnetism: preliminary results. Gupta–Bleuler quantization vs 't Hooft quantization. ‡.

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*G. 't Hooft, Erice lectures (1999); Class. Quant. Grav. (1999); *The Cellular Automaton Interpretation of Quantum Mechanics*, Springer (2016).

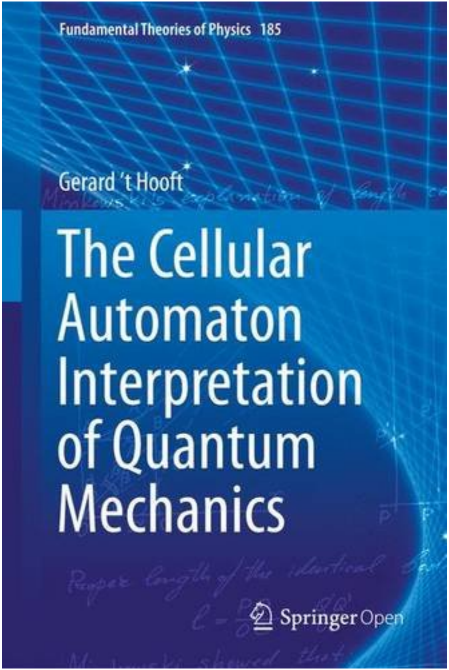
†M. B., P. Jizba and G. Vitiello, PLA (2001); M. B., F. Scardigli, P. Jizba and G. Vitiello, PLA (2009).

‡M. B., E. Celeghini, P. Jizba, F. Scardigli and G. Vitiello, arXiv:1801.06311 [quant-ph], to appear in the Proceedings of Symmetries in Science XVII.

Summary

1. Deterministic Quantum Mechanics *à la 't Hooft*
2. Dissipation and Quantization
3. Discrete models
4. "Deterministic" Electromagnetism

Deterministic Quantum Mechanics *à la 't Hooft*


Fundamental Theories of Physics 185
Gerard 't Hooft
The Cellular Automaton Interpretation of Quantum Mechanics
Springer Open

Deterministic Quantum Mechanics *à la 't Hooft*§

-Motivation: Holographic principle. problems with locality.

⇒Quantum Mechanics (QM) is not fundamental:

" the apparently quantum mechanical nature of our world is due to the statistics of fluctuations that occur at the Planck scale, in terms of a regime of completely deterministic dynamics."

- Quantum states are derived concepts, with a not strictly locally formulated definition. Their role is to make statistical predictions.

- The paradox of the holographic principle is then solved by assuming than the set of the quantum states ($\sim$ Surface) is much smaller than the set of all ontological states ($\sim$ Volume).

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§G. 't Hooft, *Determinism and dissipation in quantum gravity*, Erice lectures (1999); *Quantum gravity as a dissipative deterministic system* , Class. Quant. Grav. (1999);
The Cellular Automaton Interpretation of Quantum Mechanics, Springer (2016)

Q.: is this “hidden variables”? what about Bell’s inequalities?

A.1: most of the symmetries on which is based Bell’s theorem are absent at the Planck scale.

A.2: the definition of equivalence classes is non-local.

A.3: superdeterminism.

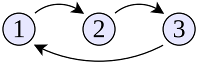
Other motivations

- problems with quantum cosmology;
- non-renormalizability of quantum gravity;
- black holes and QM.

- wish for "reality" behind QM: necessity of removing "every single bit of mysticism from quantum theory" (Copenhagen Interpretation).

Key idea: *any deterministic, time-reversible system can be described using a QM Hilbert space, where states obey a Schrödinger equation, and where the absolute squares of the coefficients of the wave functions represent probabilities.*

Example:



$$|\psi\rangle \ = \ \alpha|1\rangle + \beta|2\rangle + \gamma|3\rangle$$

Time evolution (discrete):

$$|\psi\rangle_{t+1} \ = \ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} |\psi\rangle_t = U(t, t+1)|\psi\rangle_t$$

The probabilities for being in a given state are:

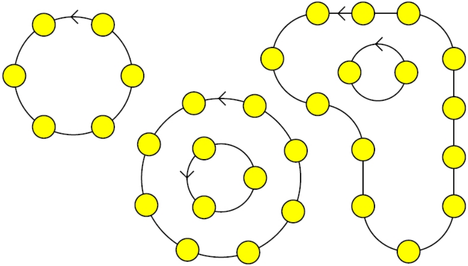
$$P(1) = |\alpha|^2; \quad P(2) = |\beta|^2; \quad P(3) = |\gamma|^2$$

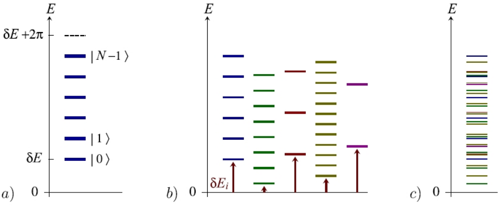
In a basis in which U is diagonal one has:

$$U = \exp(-iH\delta t); \qquad H = \begin{pmatrix} 0 & & \\ & -2\pi/3 & \\ & & -4\pi/3 \end{pmatrix}$$

$$\begin{aligned}
|0\rangle_H &= \frac{1}{\sqrt{3}} \left(|1\rangle + |2\rangle + |3\rangle\right) \\
|1\rangle_H &= \frac{1}{\sqrt{3}} \left(|1\rangle + e^{2\pi i/3}|2\rangle + e^{-2\pi i/3}|3\rangle\right) \\
|2\rangle_H &= \frac{1}{\sqrt{3}} \left(|1\rangle + e^{-2\pi i/3}|2\rangle + e^{2\pi i/3}|3\rangle\right)
\end{aligned}$$

This idea can be generalized and complicated spectra can be obtained:





Ontological states and templates

Ontological states $|A\rangle$ are describing the state a deterministic system is in. Such states form a basis for the Hilbert space: $\langle A|B\rangle = \delta_{AB}$.

Hilbert space is generated by linear combinations (superpositions) of such states. This defines general states, which are *quantum states* $|\psi\rangle$:

$$|\psi\rangle = \sum_A \lambda_A |A\rangle, \qquad \sum_A |\lambda_A|^2 = 1$$

Quantum states can be used as *templates* for doing physics:

– A template is a quantum state of the above form describing a situation in which the probability of finding the system to be in the ontological state $|A\rangle$ is $|\lambda_A|^2$.

Beables, Changeables and Superimposables

Three types of operators:

- *Beables*: operators characterizing ontological states, so they are diagonal in the ontological basis:

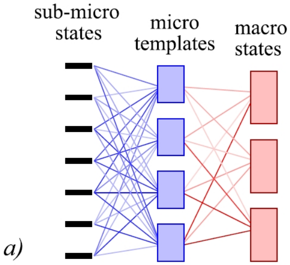
$$\mathcal{O}_{op}|A\rangle = \mathcal{O}_{op}|A\rangle. \qquad (\textit{beable})$$

- *Changeables*: operators that replace an ontological state by another ontological states, so acting like permutation operators:

$$\mathcal{O}_{op}|A\rangle = |B\rangle. \qquad (\textit{changeable})$$

- *Superimposables*: operators that map ontological states onto superpositions of ontological states

$$\mathcal{O}_{op}|A\rangle = \lambda_1|A\rangle + \lambda_2|B\rangle + \ldots \qquad (\textit{superimposables})$$


sub-micro states
micro templates
macro states
a)
sub-micro states
macro states
b)

Systems with continuous time

A quantum theory can be said to be deterministic if (in the Heisenberg picture) a complete set of operators $O_i(t)$ $(i = 1, .., N)$ exist, such that:

$$[O_i(t), O_j(t')] = 0, \quad \forall t, t'; \quad i, j = 1, .., N.$$

These operators are called "beables".

⇒ Classical systems of the form

$$H = \sum_i p_i\, f_i(q)$$

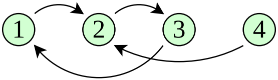
$$\begin{aligned} \dot{q}_i &= \{q_i, H\} = f_i(q)\,, \\ \dot{p}_i &= \{p_i, H\} = p_i \frac{\partial f_i(q)}{\partial q_i}\,. \end{aligned}$$

evolve deterministically even after quantization (the q_i can be regarded as beables).

Information loss

However, the above Hamiltonian is not bounded from below.
Information loss is introduced in order to get a lower bound for H.

Example:



$$|\psi\rangle \;=\; \alpha|1) + \beta|2) + \gamma|3) + \delta|4)$$

Time evolution (not unitary!):

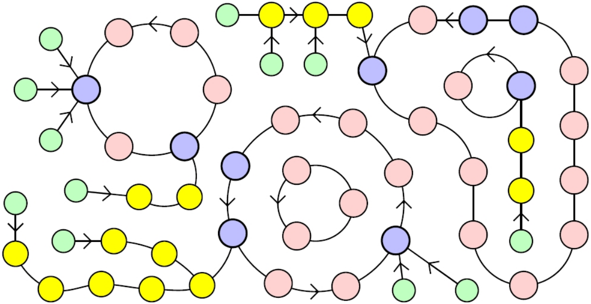
$$U_d(t+1,t) \;=\; \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The states $|1)$ and $|4)$ are equivalent, in the sense that they end up in the same state after a finite time.

Quantum states have to be identified with equivalence classes:

$$|1\rangle \equiv \{|1),|4)\}, \quad |2\rangle \equiv \{|2)\}, \quad |3\rangle \equiv \{|3)\}$$

They represent the stable orbits of the deterministic system.



Let $\rho(q)$ be a (positive) function of the q_i such that $[\rho, H] = 0$. We can then perform the split:

$$H = H_{\rm I} - H_{\rm II}$$

$$H_{\rm I} = \frac{1}{4\rho}\left(\rho + H\right)^2 \quad , \quad H_{\rm II} = \frac{1}{4\rho}\left(\rho - H\right)^2 \ .$$

$H_{\rm I}$ and $H_{\rm II}$ are positively definite and

$$[H_{\rm I}, H_{\rm II}] \, = \, [\rho, H] = 0 \, .$$

To get the lower bound for the Hamiltonian we impose the constraint:

$$H_{\rm II}|\psi\rangle = 0 \, .$$

projecting out the states which provide the negative part of the energy spectrum $\Rightarrow$ one gets rid of the unstable trajectories and $H_{\rm I}$ acquires a discrete spectrum:

$$H|\psi\rangle = H_{\rm I}|\psi\rangle = \rho|\psi\rangle \quad ; \qquad \frac{d}{dt}|\psi\rangle = -iH_{\rm I}|\psi\rangle$$

If there are stable orbits with period $T(\rho)$:

$$e^{-iHT}|\psi\rangle = |\psi\rangle \quad ; \qquad \rho\, T(\rho) = 2\pi n \, , \quad n \in \mathbb{Z}$$

Dissipation and Quantization

Dissipation and Quantization[†]

- Motivation: find specific models realizing 't Hooft idea;

- We consider a system of dissipative oscillators which has already revealed to be a useful playground for the quantization of dissipative systems[*] ;

- Our analysis seems to support 't Hooft arguments;

- Novel features: geometric phase, thermodynamical interpretation.

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[*] E.Celeghini, M.Rasetti and G.Vitiello, Ann. Phys. (1992)
[†] M. Blasone, P. Jizba and G. Vitiello, Phys. Lett. A (2001)

System of damped and amplified harmonic oscillators[‡]

$$
\begin{aligned}
m\ddot{x} + \gamma\dot{x} + \kappa x &= 0\,, \\
m\ddot{y} - \gamma\dot{y} + \kappa y &= 0\,.
\end{aligned}
$$

with Lagrangian

$$
L = m\dot{x}\dot{y} + \frac{\gamma}{2}(x\dot{y} - \dot{x}y) - kxy \quad .
$$

The canonical momenta are:

$$
p_x \equiv \frac{\partial L}{\partial \dot{x}} = m\dot{y} - \frac{\gamma}{2}y\,, \qquad p_y \equiv \frac{\partial L}{\partial \dot{y}} = m\dot{x} + \frac{\gamma}{2}x
$$

The Hamiltonian is

$$
H = \frac{1}{m}p_x p_y + \frac{\Gamma}{m}\,(y p_y - x p_x) + m\Omega^2\, xy\,,
$$

where

$$
\Gamma \equiv \gamma/2m; \qquad \Omega \equiv \sqrt{\frac{1}{m}(\kappa - \frac{\gamma^2}{4m})} \quad , \quad \kappa > \frac{\gamma^2}{4m}
$$

[‡] H. Bateman, Phys. Rev. (1931)
H. Feshbach and Y. Tikochinski, Trans. N.Y. Acad. Sci. (1977)

In the rotated coordinates:

$$x = \frac{x_1 + x_2}{\sqrt{2}} \qquad ; \qquad y = \frac{x_1 - x_2}{\sqrt{2}}$$

the Lagrangian becomes:

$$\begin{aligned} L &= L_{0,1} - L_{0,2} + \frac{\gamma}{2}(\dot{x}_1 x_2 - \dot{x}_2 x_1) \\ L_{0,i} &= \frac{m}{2}\dot{x}_i^2 - \frac{k}{2}x_i^2 \quad , \quad i = 1, 2. \end{aligned}$$

The momenta are

$$p_1 = m\dot{x}_1 + \frac{\gamma}{2}x_2 \; ; \qquad p_2 = -m\dot{x}_2 - \frac{\gamma}{2}x_1$$

Hamiltonian

$$\begin{aligned} H &= H_1 - H_2 \\ &= \frac{1}{2m}(p_1 - \frac{\gamma}{2}x_2)^2 + \frac{k}{2}x_1^2 - \frac{1}{2m}(p_2 + \frac{\gamma}{2}x_1)^2 - \frac{k}{2}x_2^2 \quad . \end{aligned}$$

Equations of motion:

$$m\ddot{x}_1 + \gamma\dot{x}_2 + kx_1 = 0 \; ; \qquad m\ddot{x}_2 + \gamma\dot{x}_1 + kx_2 = 0$$

Hyperbolic polar coordinates:

$$
\begin{aligned}
x_1 &= r\cosh u \\
x_2 &= r\sinh u
\end{aligned}
$$

The Hamiltonian becomes:

$$
H = 2\Omega\mathcal{C} - 2\Gamma J_2
$$

with

$$
\begin{aligned}
\mathcal{C} &= \frac{1}{4\Omega m}\left[\left(p_1^2 - p_2^2\right) + m^2\Omega^2\left(x_1^2 - x_2^2\right)\right] \\
&= \frac{1}{4\Omega m}\left[p_r^2 - \frac{1}{r^2}p_u^2 + m^2\Omega^2 r^2\right],
\end{aligned}
$$

$$
J_2 = \frac{m}{2}\left[(\dot{x}_1 x_2 - \dot{x}_2 x_1) - \Gamma r^2\right] = \frac{1}{2}\,p_u\,.
$$

The algebraic structure of the Hamiltonian is that of $su(1,1)$.

Let us perform the (nonlinear) canonical transformations:

$$p_1 = \mathcal{C} \qquad , \qquad q_1 = -\cot^{-1}\left[\frac{2\hat{p}_r}{\hat{p}_r^2 - \hat{p}_u^2 - 1}\right] ,$$

$$p_2 = J_2 \qquad , \qquad q_2 = 2u - \tanh^{-1}\left[\frac{\hat{p}_u^2 + \hat{p}_r^2 + 1}{2\hat{p}_r\hat{p}_u}\right] ,$$

with $\hat{p}_u \equiv \frac{p_u}{r^2 m\Omega}$ and $\hat{p}_r \equiv \frac{p_r}{rm\Omega}$.

We can then write our Hamiltonian in the 't Hooft form:

$$H \;=\; \sum_i p_i\, f_i(q) \;=\; 2\Omega\mathcal{C} - 2\Gamma J_2$$

with $f_1(q) = 2\Omega$ and $f_2(q) = -2\Gamma$.

One has $\{q_i, p_i\} = 1$, and all the other Poisson brackets vanishing.

Ladder operators:

$$A = \frac{1}{\sqrt{2\hbar m\Omega}}\left[p_1 - im\Omega x_1\right] \quad ; \quad B = \frac{1}{\sqrt{2\hbar m\Omega}}\left[p_2 - im\Omega x_2\right]$$

The Hamiltonian is

$$\begin{aligned} H &= \hbar\Omega(A^\dagger A - B^\dagger B) + i\hbar\Gamma(A^\dagger B^\dagger - AB) \\ &= 2\hbar(\Omega\mathcal{C} - \Gamma J_2)\,, \end{aligned}$$

$su(1,1)$ algebra:$[J_+, J_-] = -2J_3\,, \quad [J_3, J_\pm] = \pm J_\pm\,.$

$$\begin{aligned} \mathcal{C}^2 &= \frac{1}{4}(A^\dagger A - B^\dagger B)^2\,, \\ J_+ &= A^\dagger B^\dagger\,, \quad J_- = AB\,, \qquad J_3 = \frac{1}{2}(A^\dagger A + B^\dagger B + 1)\,, \end{aligned}$$

Denoting with $\{|n_A, n_B\rangle\}$ the set of simultaneous eigenvectors of $A^\dagger A$ and $B^\dagger B$ and setting:

$$j = \frac{1}{2}(n_A - n_B), \qquad m = \frac{1}{2}(n_A + n_B)\,,$$

we get

$$\mathcal{C}|j,m\rangle = j\,|j,m\rangle\,, \qquad J_3\,|j,m\rangle = \left(m+\frac{1}{2}\right)|j,m\rangle\,,$$

with : $|j| = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $m = |j|, |j|+\frac{1}{2}, |j|+1, \ldots$.

We can then define: $|\Psi_{j,m}\rangle \equiv \exp\left(\frac{\pi}{2}J_1\right)|j,m\rangle\,,$

$$\begin{aligned} J_2\,|\Psi_{j,m}\rangle &= i\left(m+\frac{1}{2}\right)|\Psi_{j,m}\rangle \\ \mathcal{C}\,|\Psi_{j,m}\rangle &= j\,|\Psi_{j,m}\rangle \end{aligned}$$

- Note that J_2 has a purely imaginary spectrum, although it appears to be hermitian. This is due to the choice of the (non-unitary) representation. $\Rightarrow$ modify inner product. Define "bra" vector as: $\langle\psi_{n,l}(t)| \equiv [\mathcal{T}\,|\psi_{n,l}(t)\rangle]^\dagger$.

Split of H into two positively definite parts:

$$
\begin{aligned}
H &= H_{\text{I}} - H_{\text{II}} \\
H_{\text{I}} &= \frac{1}{2\Omega\mathcal{C}}(2\Omega\mathcal{C} - \Gamma J_2)^2 \\
H_{\text{II}} &= \frac{\Gamma^2}{2\Omega\mathcal{C}} J_2^2
\end{aligned}
$$

We require $r^2 > 0$ in order for $\mathcal{C}$ to be invertible (and positive).

Impose now the constraint on the physical states $|\psi\rangle$:

$$
H_{\text{II}}|\psi\rangle = 0 \quad \Rightarrow \quad J_2|\psi\rangle = 0\,,
$$

Consequently,

$$
H|\psi\rangle = H_{\text{I}}|\psi\rangle = 2\Omega\mathcal{C}|\psi\rangle = \left(\frac{1}{2m}p_r^2 + \frac{K}{2}r^2\right)|\psi\rangle\,,
$$

where $K \equiv m\Omega^2$.

H_{I} thus reduces to the Hamiltonian for the linear (radial) harmonic oscillator $\ddot{r} + \Omega^2 r = 0$.

The generic state $|\psi\rangle_H$ can be written as

$$|\psi(t)\rangle_H = \hat{T}\left[\exp\left(\frac{i}{\hbar}\int_{t_0}^{t} 2\Gamma J_2 dt'\right)\right]|\psi(t)\rangle_{H_{\rm I}} ,$$

where $\hat{T}$ denotes time-ordering. We have:

$$\begin{aligned} i\hbar\frac{d}{dt}|\psi(t)\rangle_H &= H\,|\psi(t)\rangle_H , \\ i\hbar\frac{d}{dt}|\psi(t)\rangle_{H_{\rm I}} &= 2\Omega\mathcal{C}|\psi(t)\rangle_{H_{\rm I}} . \end{aligned}$$

We can write

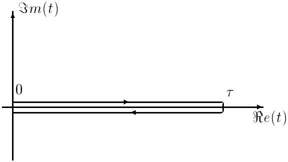
$$|\psi(t)\rangle_H = \exp\left(i\int_{C_t} A(t')dt'\right)|\psi(t)\rangle_{H_{\rm I}} ,$$

where $A(t) \equiv \frac{\Gamma m}{\hbar}(\dot{x}_1 x_2 - \dot{x}_2 x_1)$ and $\int_{C_t} r^2 = 0$.

For eigenstates of H and $H_{\rm I}$ we have

$$\begin{aligned} & {}_H\langle\psi(\tau)|\psi(0)\rangle_H \\ & ={}_{H_{\rm I}}\langle\psi(0)|\exp\left(i\int_{C_{0\tau}} A(t')dt'\right)|\psi(0)\rangle_{H_{\rm I}} \\ & \equiv e^{i\phi} , \end{aligned}$$

The contour $C_{0\tau}$ is the one going from $t' = 0$ to $t' = \tau$ and back.



The closed-time-path used for the calculation of the geometric phase.

We show that

$$\int_{C_{0\tau}} A(t')dt' = -\frac{\Gamma m}{\hbar} R^2 \,\Im\left[\int_{\Delta} \frac{dz}{z}\right] = \pi R^2 \frac{\gamma}{\hbar} \equiv \alpha\pi \ .$$

Physical states are periodical, thus

$$\begin{aligned} |\psi(\tau)\rangle &= \exp\left[i\phi - \frac{i}{\hbar}\int_0^\tau \langle\psi(t)|H|\psi(t)\rangle dt\right] |\psi(0)\rangle \\ &= \exp\left(-i2\pi n\right)|\psi(0)\rangle \,, \end{aligned}$$

i.e.

$$\frac{\langle\psi(\tau)|H|\psi(\tau)\rangle}{\hbar}\tau - \phi = 2\pi n \qquad , \quad n = 0, 1, 2, ...$$

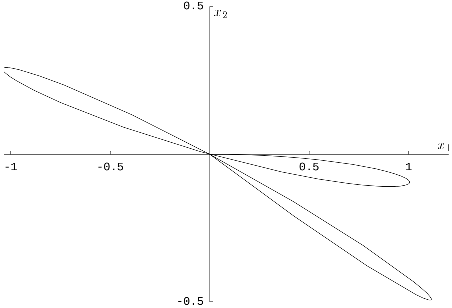
which by using $\tau = \frac{2\pi}{\Omega}$ and $\phi = \alpha\pi$, gives

$$E^n_{I,eff} = \langle\psi_n(\tau)|H|\psi_n(\tau)\rangle = \hbar\Omega\left(n + \frac{\alpha}{2}\right) \ ,$$

$E^n_{I,eff}$ denotes the effective energy of the n-th energy level of the physical system, namely the energy given by H_I corrected by its interaction with environment.

• The dissipation term J_2 of the Hamiltonian, which manifests as the geometrical phase $\phi = \alpha\pi$, is actually responsible for the $n = 0$ "zero point energy": $\mathcal{E}_0 = \hbar\Omega\frac{\alpha}{2}$.

• Setting $\alpha = 1$ gives $\Gamma = \frac{\Omega}{2}$.



Trajectories for $r_0 = 0$ and $v_0 = \Omega$, after three half-periods for $\kappa = 20$, $\gamma = 1.2$ and $m = 5$.

The ratio $\int_0^{\tau/2}(\dot{x}_1 x_2 - \dot{x}_2 x_1)dt/\mathcal{E} = \pi\frac{\Gamma}{m\Omega^3}$ is preserved.

We have (using $u = -\Gamma t$):

$$i\hbar \frac{d}{dt}|\psi(t)\rangle_H = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle_H + i\hbar \frac{du}{dt}\frac{\partial}{\partial u}|\psi(t)\rangle_H \ ,$$

The dissipation contribution to the energy is thus described by the "translations" in the u variable.

The "full Hamiltonian" H formally plays role of the free energy $\mathcal{F}$[§]:

$$H = H_{\rm I} - (\hbar\Gamma)\frac{2J_2}{\hbar} \equiv U - TS = \mathcal{F}$$

with $U \equiv H_{\rm I} = 2\Omega\mathcal{C}$, $S \equiv \frac{2J_2}{\hbar}$ and $T = \hbar\Gamma$.

$2\Gamma J_2$ represents the heat contribution in H.

It is remarkable that the "temperature" $\hbar\Gamma$ equals the zero point energy: $\hbar\Gamma = \frac{\hbar\Omega}{2}$.

[§] E.Celeghini, M.Rasetti and G.Vitiello, *Ann. Phys.* (1992)

Wave functions for the dual oscillator system*

Kernel:

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle \equiv \langle \mathbf{x}_b | U(t_b, t_a) | \mathbf{x}_a \rangle \,.$$

time evolution operator fulfils Schrödinger equations

$$\begin{aligned} i\hbar \frac{\partial}{\partial t_b} U(t_b, t_a) &= \hat{H}\, U(t_b, t_a)\,, \\ i\hbar \frac{\partial}{\partial t_b} U(t_a, t_b) &= -\,U(t_a, t_b)\, \hat{H} \quad , \quad t_b > t_a\,, \end{aligned}$$

the kernel satisfies the equations

$$\begin{aligned} i\hbar \frac{\partial}{\partial t_b} \langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle &= \hat{H}\left(-i\hbar\, \partial_{\mathbf{x}_b}, \mathbf{x}_b\right) \langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle\,, \\ i\hbar \frac{\partial}{\partial t_b} \langle \mathbf{x}_a; t_a | \mathbf{x}_b; t_b \rangle &= -\,\mathcal{T} \hat{H}\left(-i\hbar\, \partial_{\mathbf{x}_b}, \mathbf{x}_b\right) \mathcal{T}^{-1} \langle \mathbf{x}_a; t_a | \mathbf{x}_b; t_b \rangle \end{aligned}$$

with the initial condition

$$\lim_{t_b \to t_a} \langle \mathbf{x}_a; t_a | \mathbf{x}_b; t_b \rangle = \delta(\mathbf{x}_a - \mathbf{x}_b)\,.$$

*M. Blasone and P. Jizba, Annals Phys. (2004) .

If the Hamiltonian is time–independent:

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = \langle \mathbf{x}_b | \exp\left(-\frac{i}{\hbar} \hat{H}(t_b - t_a) \right) | \mathbf{x}_a \rangle \, .$$

For quadratic systems, the kernel can be written as:

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle = F[t_a, t_b] \exp\left(\frac{i}{\hbar} S_{cl}[\mathbf{x}] \right) \, , \quad t_b > t_a \, .$$

The function $F[t_a, t_b]$ is the so called *fluctuation factor* .

These two quantities can be completely expressed in terms of classical solutions:

$$
\begin{aligned}
S[\ bfx] &= \int_{t_a}^{t_b} dt\, L\,, \\
&= \frac{m}{2}[\mathbf{x}_{cl}(t_b)\dot{\mathbf{x}}_{cl}(t_b) - \mathbf{x}_{cl}(t_a)\dot{\mathbf{x}}_{cl}(t_a)]\,.
\end{aligned}
$$

and, for the Van Vleck determinant:

$$
F[t_a, t_b] = \sqrt{det_2\left(\frac{i}{2\pi\hbar}\,\frac{\partial^2 S_{cl}}{\partial \mathbf{x}_a^\alpha \partial \mathbf{x}_b^\beta}\right)} = \frac{m}{2\pi\hbar}\sqrt{\frac{W}{D}}
$$

Wronskian:

$$
W(t) = \begin{vmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 \\ \dot{\mathbf{u}}_1 & \dot{\mathbf{u}}_2 & \dot{\mathbf{u}}_3 & \dot{\mathbf{u}}_4 \end{vmatrix}, \qquad D = \begin{vmatrix} \mathbf{u}_1(t_a) & \mathbf{u}_2(t_a) & \mathbf{0} & \mathbf{0} \\ \mathbf{u}_2(t_b) & \mathbf{u}_2(t_b) & \mathbf{v}_1(t_b) & \mathbf{v}_2(t_b) \end{vmatrix}
$$

Actually the above formula is correct only for sufficiently short times $t_b - t_a$.

In the general case $F[t_a, t_b]$ will become infinite every time when the classical (position space) orbit touches (or crosses) a caustic.

A more general expression is:

$$\langle \mathbf{x}_b; t_b | \mathbf{x}_a; t_a \rangle \quad = \quad e^{-i\frac{\pi}{2} n_{a,b}} F[t_a, t_b] \exp\left(\frac{i}{\hbar} S_{cl}[\mathbf{x}]\right)$$

Here $n_{a,b}$ is the Morse index of the classical path running from $\mathbf{x}_a$ to $\mathbf{x}_b$.

The Morse index then counts how many times the classical orbit crosses (or touches) the caustic.

Explicit form of the kernel:

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \frac{m}{2\pi\,\hbar}\sqrt{\frac{W}{D}}\exp\left[-\frac{i\,m}{4D\,\hbar}\left(\frac{dD}{dt_a}\,r_a^2 - \frac{dD}{dt_b}\,r_b^2\right)\right]$$
$$\times \quad \exp\left[\frac{i\,m}{\hbar}\sqrt{\frac{W}{D}}\,r_a r_b\,\cosh(\Delta u - \alpha)\right].$$

with $\alpha(t_a, t_b) = \Gamma\,(t_a - t_b) + \beta$.

Use the formulas:

$$\exp(i\,a\cosh(u)) = \sum_{l=-\infty}^{\infty}(-1)^l\,I_l(-i\,a)\,e^{-lu}\,,$$

$$\sum_{n=0}^{\infty} n!\,\frac{L_n^l(x)L_n^l(y)b^n}{\Gamma(n+l+1)} = \frac{(xyb)^{-\frac{1}{2}l}}{1-b}\exp\left[-b\frac{x+y}{1-b}\right]\,I_l\left(2\frac{\sqrt{xyb}}{1-b}\right).$$

$$x = \frac{m}{\hbar}\sqrt{W}\,\frac{r_a^2}{\rho(t_a)}\,, \qquad y = \frac{m}{\hbar}\sqrt{W}\,\frac{r_b^2}{\rho(t_b)}\,,$$

where $r_a^2; r_b^2 > 0$ and

$$\rho(t) = \sqrt{\sum_{i<j}^{4}(\mathbf{u}_i(t)\wedge\mathbf{u}_j(t))^2}\,.$$

The kernel can be finally rewritten as:

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle =$$

$$= \frac{i}{\pi} \sum_{n,l} \frac{n! L_n^l \left(\frac{m}{\hbar} \sqrt{W}\, r_a^2 / \rho(t_a) \right) \, L_n^l \left(\frac{m}{\hbar} \sqrt{W}\, r_b^2 / \rho(t_b) \right)}{\Gamma(n+l+1)}$$

$$\times \left[b^*(t_a) b(t_b) \right]^{n+\frac{l+1}{2}} \left(\frac{m}{\hbar} \sqrt{\frac{W}{\rho(t_a)\rho(t_b)}} \right)^{l+1} (r_a r_b)^l \, e^{l(u_a - u_b + \alpha(t_a, t_b))}$$

$$\times \exp \left(\frac{m}{2\hbar} \left[\frac{i}{2} \frac{\dot{\rho}(t_b)}{\rho(t_b)} - \frac{\sqrt{W}}{\rho(t_b)} \right] r_b^2 - \frac{m}{2\hbar} \left[\frac{i}{2} \frac{\dot{\rho}(t_a)}{\rho(t_a)} + \frac{\sqrt{W}}{\rho(t_a)} \right] r_a^2 \right)$$

It satisfies the time–dependent Schrödinger eq.:

$$\left(i\hbar \frac{\partial}{\partial t_b} - \hat{H}(r_b, u_b) \right) \langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = 0 \,, \quad t_b > t_a \,,$$

where

$$\hat{H} \quad = \quad \frac{1}{2m} \left[\hat{p}_r^2 - \frac{1}{r^2} \hat{p}_u^2 + m^2 \Omega^2 r^2 \right] - \Gamma \hat{p}_u$$

Wave functions

From the above kernel we can extract the wave functions:

$$
\begin{aligned}
\psi_{n,l}(r,u,t) &= \sqrt{\frac{1}{\pi}}\sqrt{\frac{n!}{\Gamma(n+l+1)}}\left(\sqrt{\frac{m}{\hbar\,\rho(t)}}\,W^{1/4}\right)^{l+1} \\
&\times\ [\,b(t)\,]^{n+\frac{l+1}{2}}\,r^l\,L_n^l\left(\frac{m}{\hbar}\sqrt{W}\,r^2/\rho(t)\right) \\
&\times\ \exp\left(\frac{m}{2\hbar}\left[\frac{i}{2}\frac{\dot\rho(t)}{\rho(t)}-\frac{\sqrt{W}}{\rho(t)}\right]r^2\right)e^{-l(u+\Gamma t-\frac{\beta}{2})}\,,
\end{aligned}
$$

$$
\begin{aligned}
\psi^{(*)}_{n,l}(r,u,t) &= \sqrt{\frac{1}{\pi}}\sqrt{\frac{n!}{\Gamma(n+l+1)}}\left(\sqrt{\frac{m}{\hbar\,\rho(t)}}\,W^{1/4}\right)^{l+1} \\
&\times\ [\,b^*(t)\,]^{n+\frac{l+1}{2}}\,r^l\,L_n^l\left(\frac{m}{\hbar}\sqrt{W}\,r^2/\rho(t)\right) \\
&\times\ \exp\left(-\frac{m}{2\hbar}\left[\frac{i}{2}\frac{\dot\rho(t)}{\rho(t)}+\frac{\sqrt{W}}{\rho(t)}\right]r^2\right)e^{l(u+\Gamma t-\frac{\beta}{2})}\,,
\end{aligned}
$$

Radial kernel

This is defined as

$$\langle r_b, u_b; t_b | r_a, u_a; t_a \rangle = \sum_{n,l} \frac{\langle r_b; t_b | r_a; t_a \rangle_{n,l}}{\pi \sqrt{r_a r_b}} \, e^{l(\alpha(t) - \Delta u)} \, .$$

It satisfies the time–dependent Schrödinger equation

$$\left(i\hbar \frac{\partial}{\partial t_b} - \hat{H}_l(r_b) \right) \langle r_b; t_b | r_a; t_a \rangle_{n,l} = 0 \, , \quad t_b > t_a \, ,$$

$$\hat{H}_l = \frac{1}{2m} \left[-\hbar^2 \frac{\partial^2}{\partial r^2} + \frac{\hbar^2}{r^2} \left(l^2 - \frac{1}{4} \right) + m^2 \Omega^2 r^2 \right]$$

The radial wave function $\psi_{n,l}(r,t) = \langle r | \psi_{n,l}(t) \rangle$ reads

$$\begin{aligned} \psi_{n,l}(r,t) \quad &= \quad \sqrt{\frac{n!}{\Gamma(n+l+1)}} \left(\sqrt{\frac{m}{\hbar\, \rho(t)}} \, W^{1/4} \right)^{l+1} \\ &\times \quad [\, b(t)\,]^{n + \frac{l+1}{2}} \, r^{l+\frac{1}{2}} \, L_n^l \left(\frac{m}{\hbar} \sqrt{W} \, r^2 / \rho(t) \right) \\ &\times \quad \exp \left(\frac{m}{2\hbar} \left[\frac{i}{2} \frac{\dot{\rho}(t)}{\rho(t)} - \frac{\sqrt{W}}{\rho(t)} \right] r^2 \right) . \end{aligned}$$

For $l = \pm\frac{1}{2}$ the above wave function reduces to the harmonic oscillator ones (generalized Laguerre polynomials $\to$ Hermite polynomials).

$$\begin{aligned}
\psi_{n,\frac{1}{2}}(r,t) &= \frac{1}{2^{2n+1}}\sqrt{\frac{1}{n!\,\Gamma\left(n+\frac{3}{2}\right)}}\left(\sqrt{\frac{m}{\hbar\,\rho(t)}}\,W^{\frac{1}{4}}\right)^{\frac{1}{2}} \\
&\times\ [\,b(t)\,]^{n+\frac{3}{4}}\,H_{2n+1}\left(\sqrt{\frac{m}{\hbar\,\rho(t)}}\,W^{\frac{1}{4}}\,r\right) \\
&\times\ \exp\left(\frac{m}{2\hbar}\left[\frac{i}{2}\,\frac{\dot{\rho}(t)}{\rho(t)}-\frac{\sqrt{W}}{\rho(t)}\right]r^2\right) ,
\end{aligned}$$

$$\begin{aligned}
\psi_{n,-\frac{1}{2}}(r,t) &= \frac{1}{2^{2n}}\sqrt{\frac{1}{n!\,\Gamma\left(n+\frac{1}{2}\right)}}\left(\sqrt{\frac{m}{\hbar\,\rho(t)}}\,W^{\frac{1}{4}}\right)^{\frac{1}{2}} \\
&\times\ [\,b(t)\,]^{n+\frac{1}{4}}\,H_{2n}\left(\sqrt{\frac{m}{\hbar\,\rho(t)}}\,W^{\frac{1}{4}}\,r\right) \\
&\times\ \exp\left(\frac{m}{2\hbar}\left[\frac{i}{2}\,\frac{\dot{\rho}(t)}{\rho(t)}-\frac{\sqrt{W}}{\rho(t)}\right]r^2\right) .
\end{aligned}$$

The geometric phase is defined as:

$$
\begin{aligned}
e^{i\phi_{BA}} &= e^{i\phi_{tot}-i\phi_{dyn}} \\
&= \langle\psi(0)|\psi(\tau)\rangle \exp\left(i\int_0^\tau dt\,\langle\psi(t)|i\frac{d}{dt}|\psi(t)\rangle\right)
\end{aligned}
$$

We get:

$$
\begin{aligned}
\phi_{BA} &= (2n+l+1)\int_{t_i}^{t_f} dt\left(\frac{\dot\rho^2}{8\rho\sqrt{W}}+\frac{\Omega^2\rho}{2\sqrt{W}}\right) \\
&- (2n+l+1)\,(\pi\,\mathrm{ind}\,\gamma) - \frac{\pi}{2}\,n_{i,f}\,.
\end{aligned}
$$

where indγ counts the number of revolutions around the origin: $\mathrm{ind}\,\gamma = \frac{1}{2\pi i}\oint_\gamma \frac{dz}{z}$.

Observe now that:

$$
\psi_n^{lho}(r,t) = \psi_{n,-\frac{1}{2}}(r,t) = \sqrt{\pi\,r}\,\psi_{n,-\frac{1}{2}}(r,-\Gamma t+\beta/2,t)\,.
$$

and so

$$
\psi_n^{lho}(r,\tau) = \sqrt{\pi\,r}\;\psi_{n,-\frac{1}{2}}(r,u,\tau)|_{u=\beta/2-\tau\Gamma}
$$

By taking into account the periodicity of $\psi_n^{lho}(r,t)$ (period $\tau = 2\pi/\Omega$),

$$\int_0^{\tau} \langle \psi_n^{lho}(t)|\hat{H}_{-\frac{1}{2}}|\psi_n^{lho}(t)\rangle \, dt = \hbar \left(2\pi n - \phi_{BA}\right) .$$

we get the quantized energy spectrum:

$$\mathrm{E}_n^{lho} = \hbar\Omega \left(n - \frac{\phi_{AB}}{2\pi}\right) .$$

In a simple case (stationary states) the geometric phase is given only by the Morse index, which is equal to 2. Therefore:

$$\mathrm{E}_n^{lho} = \hbar\Omega \left(n + \frac{1}{2}\right) .$$

An explicit example

$$
\begin{aligned}
u_{11}(t) &= \sqrt{2}\cos(\Omega t)\ \cosh(\Gamma t)\,, & u_{12}(t) &= -\sqrt{2}\cos(\Omega t)\ \sinh(\Gamma t)\,,\\
u_{21}(t) &= \sqrt{2}\cos(\Omega t)\ \sinh(\Gamma t)\,, & u_{22}(t) &= -\sqrt{2}\cos(\Omega t)\ \cosh(\Gamma t)\,,\\
v_{11}(t) &= \sqrt{2}\sin(\Omega t)\ \cosh(\Gamma t)\,, & v_{12}(t) &= -\sqrt{2}\sin(\Omega t)\ \sinh(\Gamma t)\,,\\
v_{21}(t) &= \sqrt{2}\sin(\Omega t)\ \sinh(\Gamma t)\,, & v_{22}(t) &= -\sqrt{2}\sin(\Omega t)\ \cosh(\Gamma t)\,.
\end{aligned}
$$

Wronskian

$$
W = \begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2}\Gamma & \sqrt{2}\Omega & 0 \\ -\sqrt{2}\Gamma & 0 & 0 & -\sqrt{2}\Omega \end{vmatrix} = 4\Omega^2\,,
$$

and $D = 4\sin^2\Omega(t_b - t_a)$

Classical action:

$$
S_{cl} = \frac{m\Omega}{2\sin[\Omega(t_b - t_a)]}\left\{(r_a^2 + r_b^2)\cos[\Omega(t_b - t_a)] - 2r_a r_b \cosh[u_b - u_a - \Gamma(t_b - t_a)]\right\}\,,
$$

Fluctuation factor:

$$
F[t_a, t_b] = -\frac{m}{2\pi\hbar}\,\frac{\Omega}{\sin[\Omega(t_b - t_a)]}
$$

Wave function:

$$
\psi_{n,l}(r,u,t) = \sqrt{\frac{n!}{\pi\Gamma(n+l+1)}}\left(\frac{m\sqrt{\Omega}}{\hbar}\right)^{l+\frac{1}{2}} r^l\, e^{-\frac{m\Omega}{2\hbar}r^2}\, L_n^l\left(\frac{m\Omega}{\hbar}r^2\right)\, e^{-l(u+\Gamma t)}\, e^{-i\Omega(2n+l+1)t}\,.
$$

Radial wave function:

$$
\psi_{n,l}(r,t) = \sqrt{\frac{n!}{\pi\Gamma(n+l+1)}}\left(\frac{m\sqrt{\Omega}}{\hbar}\right)^{l+\frac{1}{2}} r^{l+\frac{1}{2}}\, e^{-\frac{m\Omega}{2\hbar}r^2}
$$

Composite system*

Consider two Bateman's oscillators, labeled by the index $i = A, B$:

$$m_i \ddot{x}_i + \gamma_i \dot{x}_i + \kappa_i x_i = 0\,,$$

$$m_i \ddot{y}_i - \gamma_i \dot{y}_i + \kappa_i y_i = 0\,,$$

where $m_i = (m_A, m_B)$, $\gamma_i = (\gamma_A, \gamma_B)$ and $\kappa_i = (\kappa_A, \kappa_B)$. conjugated momenta are

$$\begin{aligned} p_{x_i} &= \frac{\partial \mathcal{L}_i}{\partial \dot{x}_i} = m_i \dot{y}_i - \frac{1}{2}\,\gamma_i y_i\,, \\ p_{y_i} &= \frac{\partial \mathcal{L}_i}{\partial \dot{y}_i} = m_i \dot{x}_i + \frac{1}{2}\,\gamma_i x_i\,. \end{aligned}$$

Hamiltonian for ith oscillator:

$$H_i = \frac{1}{m_i} p_{x_i} p_{y_i} + \frac{\gamma_i}{2m_i}(y_i p_{y_i} - x_i p_{x_i}) + \left(\kappa_i - \frac{\gamma_i^2}{4m_i}\right) x_i y_i\,.$$

---

*M. Blasone, P. Jizba, F. Scardigli and G. Vitiello, Phys. Lett. A (2009)

The algebraic structure for the total system $H_T = H_A + H_B$ is the one of $su(1,1) \otimes su(1,1)$. Indeed, from the dynamical variables $p_{\alpha i}$ and $x_{\alpha i}$ one may construct the functions

$$\begin{aligned} J_{1i} &= \frac{1}{2m_i\Omega_i}\, p_{1i}p_{2i} - \frac{m_i\Omega_i}{2}\, x_{1i}x_{2i}\,, \\ J_{2i} &= \frac{1}{2}(p_{1i}x_{2i} + p_{2i}x_{1i})\,, \\ J_{3i} &= \frac{1}{4m_i\Omega_i}\left(p_{1i}^2 + p_{2i}^2\right) + \frac{m_i\Omega_i}{4}\, \left(x_{1i}^2 + x_{2i}^2\right)\,, \end{aligned}$$

where $\Omega_i = \sqrt{\frac{1}{m_i}(\kappa_i - \frac{\gamma_i^2}{4m_i})}$, and $\kappa_i > \frac{\gamma_i^2}{4m_i}$. Applying now the canonical Poisson brackets $\{x_{\alpha i}, p_{\beta j}\} = \delta_{\alpha\beta}\delta_{ij}$we obtain the Poisson's subalgebra

$$\{J_{2i}, J_{3i}\} = J_{1i}\,, \quad \{J_{3i}, J_{1i}\} = J_{2i}\,, \quad \{J_{1i}, J_{2i}\} = -J_{3i}\,, \quad \{J_{\alpha i}, J_{\beta j}\}|_{i\neq j} = 0\,.$$

The quadratic Casimirs for the algebra are defined as

$$\mathcal{C}_i^2 = J_{3i}^2 - J_{2i}^2 - J_{1i}^2 \,.$$

The $\mathcal{C}_i$ explicitly read

$$\mathcal{C}_i = \frac{1}{4m_i\Omega_i}\left[(p_{1i}^2 - p_{2i}^2) + m_i^2\Omega_i^2(x_{1i}^2 - x_{2i}^2)\right].$$

In terms of J_{2i} and $\mathcal{C}_i$ the Hamiltonians H_i are given by

$$H_i \;=\; 2\left(\Omega_i\mathcal{C}_i - \Gamma_i J_{2i}\right) \,,$$

where $\Gamma_i = \gamma_i/2m_i$.

Following 't Hooft, we now write the above Hamiltonians in the form

$$\hat{H}_i = \hat{H}_{i+} - \hat{H}_{i-}\,,$$
$$\hat{H}_{i+} = \frac{1}{4\hat{\rho}_i}(\hat{\rho}_i + \hat{H}_i)^2, \quad \hat{H}_{i-} = \frac{1}{4\hat{\rho}_i}(\hat{\rho}_i - \hat{H}_i)^2\,.$$

Choosing $\hat{\rho}_i = 2\Omega_i\hat{\mathcal{C}}_i$, and taking $\hat{\mathcal{C}}_i > 0$ (this can be done, because $\hat{\mathcal{C}}_i$ are constants of motion), the splitting reads

$$\hat{H}_{i+} = \frac{(\hat{H}_i + 2\Omega_i\hat{\mathcal{C}}_i)^2}{8\Omega_i\hat{\mathcal{C}}_i} = \frac{1}{2\Omega_i\hat{\mathcal{C}}_i}(2\Omega_i\hat{\mathcal{C}}_i - \Gamma_i\hat{J}_{2i})^2\,,$$
$$\hat{H}_{i-} = \frac{(\hat{H}_i - 2\Omega_i\hat{\mathcal{C}}_i)^2}{8\Omega_i\hat{\mathcal{C}}_i} = \frac{1}{2\Omega_i\hat{\mathcal{C}}_i}\Gamma_i^2\hat{J}_{2i}^2\,.$$

Quantization emerges after the information loss condition is imposed locally, i.e. separately on each of the Bateman oscillator:

$$\hat{J}_{2i}|\psi\rangle_{phys} = 0\,,$$

which defines/selects the physical states and is equivalent to

$$\hat{H}_{i-}|\psi\rangle_{phys} = 0, \quad i = A, B\,.$$

This implies

$$
\begin{aligned}
\hat{H}_i|\psi\rangle_{phys} &= (\hat{H}_{i+} - \hat{H}_{i-})|\psi\rangle_{phys}\,, \\
&= \hat{H}_{i+}|\psi\rangle_{phys} = 2\Omega_i\hat{\mathcal{C}}_i|\psi\rangle_{phys}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
2\Omega_i\hat{\mathcal{C}}_i|\psi\rangle_{phys} &= \left[\frac{1}{2m_i}\left(\hat{p}^2_{r_i} + m_i^2\Omega_i^2\hat{r}_i^2\right) - \frac{2\hat{J}^2_{2i}}{m_i\hat{r}_i^2}\right]|\psi\rangle_{phys}, \\
&= \left(\frac{\hat{p}^2_{r_i}}{2m_i} + \frac{m_i}{2}\Omega_i^2\hat{r}_i^2\right)|\psi\rangle_{phys}\,.
\end{aligned}
$$

Thus we obtain, for each one of the systems A and B separately, a genuine QM oscillator.

On the other hand, by writing the total Hamiltonian as

$$\begin{aligned} H_T &= 2\Omega\mathcal{C} - 2\Gamma J\,, \\ &= 2(\Omega_A\mathcal{C}_A + \Omega_B\mathcal{C}_B) - 2(\Gamma_A J_{2A} + \Gamma_B J_{2B})\,. \end{aligned}$$

with $\mathcal{C}_A,\ \mathcal{C}_B > 0 \ \Rightarrow\ \mathcal{C} > 0\,.$

$$\begin{aligned} H_+ &= \frac{(H_T + 2\Omega\mathcal{C})^2}{8\Omega\mathcal{C}} = \frac{1}{2\Omega\mathcal{C}}(2\Omega\mathcal{C} - \Gamma J)^2\,, \\ H_- &= \frac{(H_T - 2\Omega\mathcal{C})^2}{8\Omega\mathcal{C}} = \frac{1}{2\Omega\mathcal{C}}\Gamma^2 J^2\,. \end{aligned}$$

$$\hat{H}_-|\psi\rangle_{phys} = \hat{J}|\psi\rangle_{phys} = 0\,.$$

This implies appearance of nonlinear terms:

$$\hat{H}_T \approx \hat{H}_+ \approx 2\Omega\hat{\mathcal{C}}\,, \qquad \hat{J}_{2B} \approx -\frac{\Gamma_A}{\Gamma_B}\hat{J}_{2A}\,,$$

$$\hat{H}_T \approx \left(\frac{\hat{p}^2_{r_A}}{2m_A} - \frac{2\hat{J}^2_{2A}}{m_A\,\hat{r}^2_A} + \frac{1}{2}m_A\,\Omega^2_A\hat{r}^2_A\right) + \left(\frac{\hat{p}^2_{r_B}}{2m_B} + \frac{1}{2}m_B\,\Omega^2_B\hat{r}^2_B\right) - \frac{2}{m_B}\,\frac{\Gamma^2_A}{\Gamma^2_B}\,\frac{\hat{J}^2_{2A}}{\hat{r}^2_B}$$

Other dissipative systems*

Consider equation for d.h.o.

$$\ddot{x} \; + \; \gamma \dot{x} \; + \; \omega^2 x \; = \; 0$$

and the canonical variables (expanding coordinate)

$$\hat{Q} \; = \; x \, e^{\frac{\gamma}{2} t} \quad , \quad \hat{P} \; = \; m \, \dot{\hat{Q}} \; = \; m \left(\dot{x} + \frac{\gamma}{2} x \right) e^{\frac{\gamma}{2} t} \; .$$

One gets

$$\hat{H}_{exp} \; = \; \frac{1}{2m} \, \hat{P}^2 \; + \; \frac{m}{2} \Omega^2 \hat{Q}^2$$

which is a constant of motion providing the equation of motion

$$\ddot{\hat{Q}} \; + \; \Omega^2 \, \hat{Q} \; = \; 0$$

*D. Schuch and M. Blasone, J.Phys.Conf.Ser. (2017)

In terms of the physical variables x and $p = m\dot{x}$, the Hamiltonian is

$$\hat{H}_{exp} \hat{=} \frac{m}{2}\left[\dot{x}^2 \ + \ \gamma \dot{x} x \ + \ \omega^2 x^2\right] e^{\gamma t} \ = \ \text{const.} \ ,$$

Identifying, up to a constant, this Hamiltonian with the Bateman one,

$$\hat{H}_{\rm B} \hat{=} \hat{H}_{exp} \quad ,$$

we get

$$\hat{H}_{exp} \hat{=} \frac{m}{2} e^{\gamma t}\left[\dot{x}^2 \ + \ \gamma \dot{x} x \ + \ \omega^2 x^2\right] \ = \ \hat{H}_{\rm B} \hat{=} p_x \dot{x} + m\frac{\gamma}{2}\, y\dot{x} + m\omega^2 x y \, .$$

For the particular choice $c = 0$, leading to $a = \frac{m}{2}$ and $b = m\frac{\gamma}{4}$, one obtains for p_x and y

$$\hat{p}_x \; = \; \frac{m}{2}\left(\dot{x} + \frac{\gamma}{2}\; x\right) e^{\gamma t} \; = \; \frac{1}{2}\; \hat{P}\; e^{\frac{\gamma}{2} t} \quad \text{and} \quad \hat{y} = \frac{1}{2} x \; e^{\gamma t} \; = \; \frac{1}{2} \hat{Q}\; e^{\frac{\gamma}{2} t} \; .$$

Inserting this into $\hat{H}_B$ yields

$$\hat{H}_{\mathrm{B}} \; = \; \frac{1}{m}\; p_x \; p_y \; + \; m\left(\omega^2 - \frac{\gamma^2}{4}\right) x \; y \; = \; \hat{H}_\Omega$$

$$\hat{D} = \frac{\gamma}{2}\; (y \; p_y \; - \; x \; p_x) \; = \; 0 \; .$$

Expressing $\hat{D}$ in terms of x, y, $\dot{x}$ and $\dot{y}$ leads to

$$\hat{D} = \frac{m}{2}\gamma(\dot{x}y - x\dot{y}) + \frac{m}{2}\gamma^2 xy \; , \quad \text{i.e.,} \quad \hat{D} = \gamma \mathcal{J}_2 \quad .$$

Therefore, the constraint $c = 0$ leading to $\hat{D} = 0$ is equivalent to the constraint $\mathcal{J}_2 = 0$. Consequently, $\hat{H}_{exp}$ is equivalent to $\hat{H}_I$ of the split Bateman Hamiltonian,

$$\hat{H}_{exp} \; = \; \frac{1}{2m} \; \hat{P}^2 \; + \; \frac{m}{2}\Omega^2 \hat{Q}^2 = \hat{H}_{\mathrm{I}} = \frac{1}{2m} p_r^2 + \frac{m}{2}\Omega^2 r^2$$

provided the following relations are fulfilled:

$$r = x \; e^{\frac{\gamma}{2} t} = \hat{Q} \quad , \quad p_r = m \left(\dot{x} + \frac{\gamma}{2} x \right) e^{\frac{\gamma}{2} t} = \hat{P}.$$

That means the dissipative system can be described within the canonical formalism but the price is a $non-canonical$ transformation between the *physical* variables (x, p) and the *canonical* ones $(\hat{Q} = r, \hat{P} = p_r)$.

Caldirola–Kanai Lagrangian†

Caldirola and Kanai proposed the explicitly time-dependent Lagrangian

$$\hat{L}_{\rm CK} \;=\; \left[\frac{m}{2}\dot{x}^2 - V\left(x\right)\right] e^{\gamma t}$$

with the *canonical* momentum

$$\hat{p} = \frac{\partial}{\partial \dot{x}}\;\hat{L}_{\rm CK} = m\dot{x}\;e^{\gamma t} \;=\; p\;e^{\gamma t}$$

The Hamiltonian reads

$$\hat{H}_{\rm CK} \;=\; \frac{1}{2m} e^{-\gamma t}\;\hat{p}^2 + \frac{m}{2}\omega^2 x^2 e^{\gamma t}\;. \tag{1}$$

---

† P.Caldirola *Nuo. Cim.* (1941); E.Kanai *Progr. Theor. Phys.* (1948)

The Hamiltonian $\hat{H}_{\text{CK}}$ is explicitly time-dependent, *not* a constant of motion and not equivalent to the energy of the dissipative system but related to it via

$$\hat{H}_{\text{CK}} = \hat{H}_{\text{CK}}(t) = E\ e^{\gamma t}\ .$$

C-K and expanding coordinates are connected via canonical transformation:

$$\hat{Q}\ =\ \hat{x}\ e^{\frac{\gamma}{2}t}\ \ ,\ \ \hat{P}\ =\ \hat{p}\ e^{-\frac{\gamma}{2}t} + m\ \frac{\gamma}{2}\hat{x}\ e^{\frac{\gamma}{2}t}\ .$$

The explicitly time-dependent generating function $\hat{F}_2(\hat{x}, \hat{P}, t)$ connecting the corresponding Hamiltonians via

$$\hat{H}_{\text{exp}}\ =\ \hat{H}_{\text{CK}}\ +\ \frac{\partial}{\partial t}\hat{F}_2$$

is given by

$$\hat{F}_2(\hat{x}, \hat{P}, t)\ =\ \hat{x}\hat{P}\ e^{\frac{\gamma}{2}t}\ -\ m\frac{\gamma}{4}\hat{x}^2\ e^{\frac{\gamma}{2}t}\ ,$$

turning the time-dependent Hamiltonian $\hat{H}_{\text{CK}}$ into the constant of motion $\hat{H}_{\text{exp}}$.

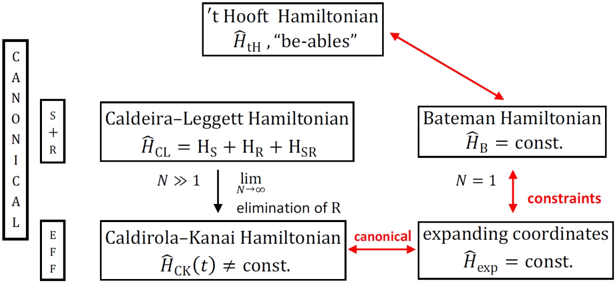
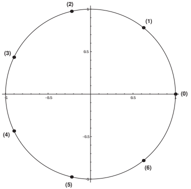


Figure 1: Relations between different descriptions of dissipative systems on the canonical level.

Discrete models

Particle on the circle and the quantum oscillator*



’t Hooft’s deterministic system for $N = 7$.

Deterministic system consisting of N states,
$\{(\nu)\} \equiv \{(0), (1), ...(N-1)\}$, on a circle:

$$(0) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}; (1) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \ldots; (N-1) = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix},$$

and $(0) \equiv (N)$.

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*G. ’t Hooft, [hep-th/0104080]; [hep-th/0105105];

Discrete time evolution:

$$t \to t + \tau \quad : \quad (\nu) \to (\nu + 1 \mod N)$$

- Finite dimensional representation $D_N(T_1)$ of the translation group.

On the basis spanned by the states (ν), the evolution operator is introduced as $(\hbar = 1)$:

$$U(\Delta t = \tau) \,=\, e^{-iH\tau} \,=\, e^{-i\frac{\pi}{N}} \begin{pmatrix} 0 & & & & & 1 \\ 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & & \ddots & \ddots & \\ & & & & 1 & 0 \end{pmatrix}$$

- The phase factor $e^{-i\frac{\pi}{N}}$ is introduced by hand.

This matrix satisfies the condition $U^N = -\mathbb{1}$ and it can be diagonalized as:

$$S\,U\,S^{-1} \;=\; e^{-i\,\frac{\pi}{N}} \begin{pmatrix} 1 & & & & \\ & e^{-i\,\frac{2\pi}{N}} & & & \\ & & e^{-i\,\frac{2\pi}{N}\,2} & & \\ & & & \ddots & \\ & & & & e^{-i\,\frac{2\pi}{N}\,(N-1)} \end{pmatrix}$$

The eigenstates of H are denoted by $|n\rangle$:

$$|n\rangle = \sum_{k=0}^{N-1} e^{-i\,\frac{2\pi n}{N}\,k}(k) \quad ; \quad n = 0, 1, ..., N-1.$$

and the spectrum is

$$H\;|n\rangle \;\;=\;\; \omega\,(n + \frac{1}{2})\;|n\rangle\;, \qquad \omega \equiv \frac{2\pi}{N\tau}.$$

- It seems to have the same spectrum as the harmonic oscillator. However its eigenvalues have an upper bound implied by the finite N value.

The $SU(2)$ description

Let us put

$$N \equiv 2l+1 \ , \quad n \equiv m+l \ , \qquad m \equiv -l, ..., l \ ,$$

We introduce the notation $|l,m\rangle$ for the states $|n\rangle$ and the operators $L_\pm$ and L_3:

$$\frac{H}{\omega} \ |l,m\rangle \ = \ (L_3 + l + \frac{1}{2}) \ |l,m\rangle = (n+\frac{1}{2}) \ |l,m\rangle \, .$$

and

$$\begin{aligned} L_3 \ |l,m\rangle &= m \ |l,m\rangle \, , \\ L_+ \ |l,m\rangle &= \sqrt{(2l-n)(n+1)} \ |l,m+1\rangle \, , \\ L_- \ |l,m\rangle &= \sqrt{(2l-n+1)n} \ |l,m-1\rangle \, . \end{aligned}$$

$su(2)$ algebra ($L_\pm \equiv L_1 \pm iL_2$):

$$[L_i, L_j\,] = i\epsilon_{ijk} L_k \ , \quad i,j,k = 1,2,3.$$

One can then introduce the analogues of position and momentum operators:

$$\hat{x} \equiv \alpha L_x, \quad \hat{p} \equiv \beta L_y, \quad \alpha \equiv \sqrt{\frac{\tau}{\pi}}, \quad \beta \equiv \frac{-2}{2l+1}\sqrt{\frac{\pi}{\tau}},$$

satisfying the "deformed" commutation relations

$$[\hat{x}, \hat{p}\,] \;=\; \alpha\beta i L_z \;=\; i\left(1 - \frac{\tau}{\pi}H\right) \,.$$

The Hamiltonian is then rewritten as

$$H \;=\; \frac{1}{2}\omega^2\hat{x}^2 \;+\; \frac{1}{2}\hat{p}^2 \;+\; \frac{\tau}{2\pi}\left(\frac{\omega^2}{4} + H^2\right) \,.$$

- Continuum limit: $l \to \infty$ and $\tau \to 0$ with ω fixed.

$\Rightarrow$ Hamitonian goes to the one of the harmonic oscillator;

$\Rightarrow$ $[\hat{x}, \hat{p}] \to 1$ and the Weyl-Heisenberg algebra $h(1)$ is obtained.

$\Rightarrow$ the original state space (finite N) changes becoming infinite dimensional.

- The above limiting procedure is nothing but a group contraction[†].

Define $a^\dagger \equiv L_+/\sqrt{2l}$, $a \equiv L_-/\sqrt{2l}$ and restore the $|n\rangle$ notation ($n = m + l$) for the states:

$$\begin{aligned} \frac{H}{\omega}\,|n\rangle &= (n + \frac{1}{2})\,|n\rangle \\ a^\dagger\,|n\rangle &= \sqrt{\frac{(2l-n)}{2l}}\sqrt{n+1}\,|n+1\rangle\,, \\ a\,|n\rangle &= \sqrt{\frac{2l-n+1}{2l}}\sqrt{n}\,|n-1\rangle\,. \end{aligned}$$

[†]M. Blasone, E. Celeghini, P. Jizba and G. Vitiello, PLA (2003);

The continuum limit is then the contraction $l \to \infty$ (fixed ω):

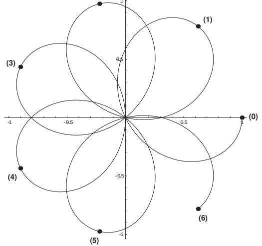
$$\begin{aligned} \frac{H}{\omega}\,|n\rangle &= (n+\frac{1}{2})\,|n\rangle\,. \\ a^{\dagger}\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle\,, \\ a\,|n\rangle &= \sqrt{n}\,|n-1\rangle\,, \end{aligned}$$

and, by inspection,

$$\begin{aligned} [a,a^{\dagger}]\,|n\rangle &= |n\rangle \\ \{a^{\dagger},a\}\,|n\rangle &= 2(n+1/2)\,|n\rangle. \end{aligned}$$

We thus have $[a,a^{\dagger}]=1$ and $H/\omega = \frac{1}{2}\{a^{\dagger},a\}$ on the representation $\{|n\rangle\}$.

- The Hilbert space, originally finite dimensional, becomes infinite dimensional under the contraction limit. Then we are led to consider an alternative model where the Hilbert space is not modified in the continuum limit.



't Hooft's deterministic system for $N = 7$.

't Hooft system recovered with underlying continuous dynamics:

$$\begin{aligned} x(t) &= \cos(\alpha t)\cos(\beta t) \\ y(t) &= -\cos(\alpha t)\sin(\beta t) \end{aligned}$$

- At the times $t_j = j\pi/\alpha$ the trajectory touches the external circle and thus π/α is the frequency of the discrete ('t Hooft) system.

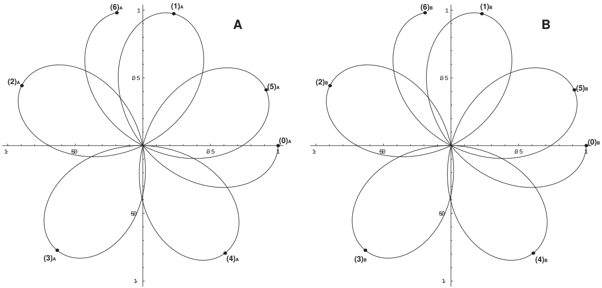
- At time t_j, the angle of $R(t_j)$ with the positive x axis is given by: $\theta_j = j\pi - \beta t_j = j(1 - \beta/\alpha)\pi$.

- When β/α is a rational number $q = M/N$, the system returns to the origin after N steps.

A deterministic system based on $SU(1,1)$

Two particles moving along two circles in discrete equidistant (synchronized) jumps. The ratio (circumference)/(length of the elementary jump) is an irrational number $\Rightarrow$ the particles never come back to the original positions:

$$t \to t + \tau \quad ; \qquad (0)_A \to (1)_A \to (2)_A \to (3)_A \ldots ,$$
$$(0)_B \to (1)_B \to (2)_B \to (3)_B \ldots .$$



Actual states (positions) can be represented by vectors with an infinite number of components.

-The one–time–step evolution operator acts on $(n)_A \otimes (m)_B$ and in the representation space of the states it reads

$$
\begin{aligned}
U(\tau) \quad &\equiv \quad e^{-iH\tau} = e^{-iH_A\tau} \otimes e^{-iH_B\tau} \\
&= \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \end{pmatrix}_A \otimes \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \end{pmatrix}_B .
\end{aligned}
$$

We work with finite dimensional matrices of dimension M and at the end of the computations perform the limit $M \to \infty$.

Define $\zeta = (1 - M)/M$. The energy eigenvectors are:

$$
|n_A\rangle = \sum_{l=0}^{M-1} e^{-i2\pi\zeta n_A l} (l)_A \; ; \;\; |n_B\rangle = \sum_{l=0}^{M-1} e^{-i2\pi\zeta n_B l} (l)_B \, .
$$

We have:

$$
U_A(\tau)|n_A\rangle = e^{i2\pi\zeta n_A} |n_A\rangle \; ; \;\; U_B(\tau)|n_B\rangle = e^{i2\pi\zeta n_B} |n_B\rangle \, .
$$

Defining $(n_A - n_B)/2 = j$ and $(n_A + n_B)/2 = m$ we may pass to the $|j, m\rangle$ basis:

$$|j, m\rangle = \sum_{l,k=0}^{M-1} e^{-i2\pi\zeta[m(k+l)+j(k-l)]} (k)_A \otimes (l)_B \,.$$

Finally, in the $M \to \infty$ limit we have:

$$\frac{H}{2}|j, m\rangle = \frac{H_A + H_B}{2}|j, m\rangle = \omega m|j, m\rangle$$

$$\frac{(H_A - H_B)}{2}\,|j, m\rangle = \omega j|j, m\rangle.$$

We then set $\mathcal{C} \equiv (H_A - H_B)/2\omega$ and $L_3 \equiv \frac{H}{\omega} + \frac{1}{2}$ and obtain the $SU(1,1)$ structure.

We can also define $L_\pm$ as:

$$L_+ \propto e^{-i2\pi(N_A + N_B)} \;;\;\; L_- \propto e^{i2\pi(N_A + N_B)} \,,$$

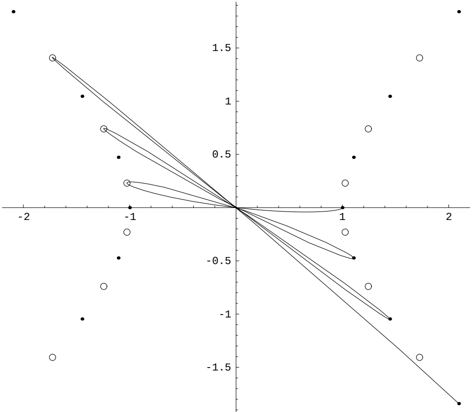
where N_A and N_B are the position operators on the circles:

$$N_A(n)_A = n(n)_A \;, \;;\;\; N_B(k)_B = k(k)_B \;,$$

Other $SU(1,1)$ deterministic systems

- a single particle "jumping" on a 2D torus. If φ_1 and φ_2 are angular coordinates (longitude and latitude) on the 2D torus and α_1/α_2 is irrational then the positions (states) never return back into the original configuration at any finite time but instead they fill up all the torus surface.

- the system of damped-amplified harmonic oscillators



From $SU(1,1)$ to $h(1)$

We have now

$$\begin{aligned} L_3|n\rangle &= (n+k)|n\rangle\,, \\ L_+|n\rangle &= \sqrt{(n+2k)(n+1)}|n+1\rangle\,, \\ L_-|n\rangle &= \sqrt{(n+2k-1)n}|n-1\rangle\,, \end{aligned}$$

where, like in $h(1)$, $n \geq 0$ is an integer and the highest weight $k > 0$ is integer or half-integer. We set

$$H/\omega = L_3 - k + 1/2\ , \quad a^\dagger = L_+/\sqrt{2k}\ , \quad a = L_-/\sqrt{2k}.$$

The $SU(1,1)$ contraction $k \to \infty$ again recovers the quantum oscillator, i.e. the $h(1)$ algebra.

- The contraction $k \to \infty$ does not modify L_3 and its spectrum but only the matrix elements of $L_\pm$.

- While in the $SU(2)$ case the Hilbert space gets modified in the contraction limit, in the $SU(1,1)$ case this does not happen.

- The $SU(2)$ model considered above says nothing about the inclusion of the phase factor.

- The $SU(1,1)$ setting, with $H = \omega L_3$, always implies a non-vanishing phase, since $k > 0$. In particular, the fundamental representation has $k = 1/2$ and thus

$$\begin{aligned} L_3|n\rangle &= (n+1/2)|n\rangle\,, \\ L_+|n\rangle &= (n+1)|n+1\rangle\,, \\ L_-|n\rangle &= n|n-1\rangle\,. \end{aligned}$$

We note that the rising and lowering operator matrix elements do not carry the square roots, as on the contrary happens for $h(1)$.

Then we introduce the following mapping in the universal enveloping algebra of $su(1,1)$:

$$a = \frac{1}{\sqrt{L_3 + 1/2}}\, L_- \quad ; \quad a^\dagger = L_+ \, \frac{1}{\sqrt{L_3 + 1/2}}$$

which gives us the wanted $h(1)$ structure, with $H = \omega L_3$.

- no limit (contraction) is necessary!

- one-to-one (non-linear) mapping between the deterministic $SU(1,1)$ system and the quantum harmonic oscillator.

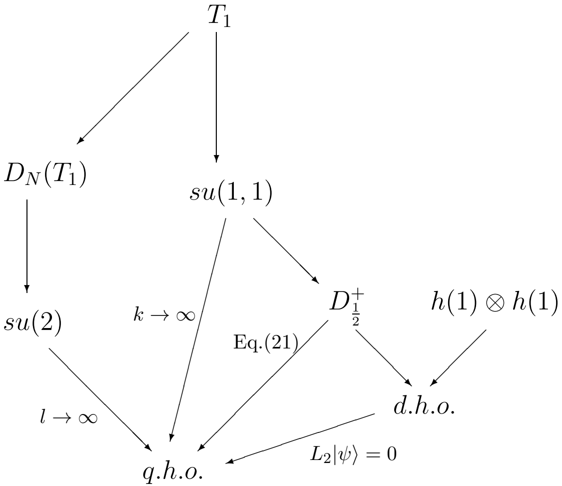
- Non-compact analog of the well-known Holstein-Primakoff representation for $SU(2)$ spin systems[‡].

- The 1/2 term in the L_3 eigenvalues now is implied by the representation.

- After a period $T = 2\pi/\omega$, the evolution of the state presents a phase π that it is not of dynamical origin ($e^{-iHT} \neq 1$): it is a geometric-like phase related to the isomorphism between $SO(2,1)$ and $SU(1,1)/Z_2$ ($e^{i2\times 2\pi L_3} = 1$)

[‡]T. Holstein and H. Primakoff, Phys. Rev. (1940).
C. C. Gerry, J. Phys. A (1983).

A schematic representation of the different quantization routes explored.



“Deterministic” Electromagnetism

Gupta–Bleuler quantization of EM field[†]

Canonical quantization of the Maxwell field in the Lorenz gauge requires the introduction of a gauge fixing term leading to the Fermi Lagrangian density[*]

$$\mathcal{L} \;=\; -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\zeta\left(\partial_\mu A^\mu\right)^2$$

Equations of motions are

$$\Box A^\mu - (1-\zeta)\partial_\mu\left(\partial_\sigma A^\sigma\right) \;=\; 0$$

If we restrict to the case $\zeta = 1$ (Feynman gauge), Lagrangian and equations of motion assume the simple form:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu \\ \Box A^\mu &= 0 \end{aligned}$$

[*] E.Fermi, (1932).

[†] S.Gupta, Proc. Roy. Soc. (1950); K.Bleuler, Helv.Phys.Acta (1950).

By introducing the conjugate momenta

$$\pi_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A^\mu)} = -\partial^0 A^\mu$$

we obtain the Hamiltonian density

$$\begin{aligned}\mathcal{H} &= -\frac{1}{2}\pi_\mu \pi^\mu + \frac{1}{2}\partial_k A_\nu \partial^k A^\nu \\ &= \frac{1}{2}\sum_{k=1}^{3}\left[(\dot{A}^k)^2 + (\nabla A^k)^2\right] - \frac{1}{2}\left[(\dot{A}^0)^2 + (\nabla A^0)^2\right]\end{aligned}$$

not positive definite!

Fourier expansion of the A^μ field:

$$A^\mu(x) = \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}} \sum_{\lambda=0}^{3} \left(a_{\mathbf{k}\lambda}\epsilon^\mu(\mathbf{k},\lambda)e^{-ik\cdot x} + a^*_{\mathbf{k}\lambda}\epsilon^\mu(\mathbf{k},\lambda)e^{ik\cdot x}\right)$$

Quantization is achieved by imposing commutation relations for the field operators A^μ and π^μ:

$$[A^\mu(\mathbf{x},t), \pi^\nu(\mathbf{y},t)] = i g^{\mu\nu} \delta^3(\mathbf{x}-\mathbf{y})$$

Commutation relations for the ladder operators:

$$\left[a_{\mathbf{k}',\lambda'}, a^\dagger_{\mathbf{k},\lambda}\right] = -g_{\lambda\lambda'} \delta^3(\mathbf{k}'-\mathbf{k})$$

Wrong sign for scalar photons $\Rightarrow$ *negative norm states.*

The Hamiltonian becomes

$$H = \int d^3\mathbf{k}\, \omega_k \left(\sum_{\lambda=1,3} a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} - a^\dagger_{\mathbf{k},0} a_{\mathbf{k},0} \right)$$

Lorenz gauge condition cannot be enforced at operatorial level, but only on the (physical) states (Gupta–Bleuler condition):

$$\partial^\mu A_\mu^{(+)}|\Phi\rangle = 0$$

or, equivalently,

$$\widehat{L}_{\mathbf{k}}|\Phi\rangle = (a_{\mathbf{k},0} - a_{\mathbf{k},3})|\Phi\rangle = 0$$

which implies that physical states $|\Phi\rangle$ should contain an equal number of longitudinal and scalar photons:

$$\langle\Phi|a^\dagger_{\mathbf{k},0} a_{\mathbf{k},0}|\Phi\rangle = \langle\Phi|a^\dagger_{\mathbf{k},3} a_{\mathbf{k},3}|\Phi\rangle$$

In this way, negative norm states are eliminated and Hamiltonian is positive definite:

$$\langle\Phi|H|\Phi\rangle = \int d^3\mathbf{k}\,\omega_k \sum_{\lambda=1,2} n_{\mathbf{k},\lambda}$$

In the GB construction, the physical states can be generated from the purely transverse states $|\Phi_T\rangle$ in the following way:

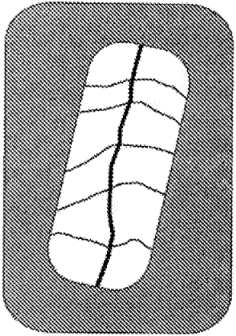
$$|\Phi\rangle = R_c|\Phi_T\rangle$$

where

$$R_c = 1 + \int d^3k c(\mathbf{k})\widehat{L}^\dagger_{\mathbf{k}} + \int d^3k d^3k' c(\mathbf{k})c(\mathbf{k}')\widehat{L}^\dagger_{\mathbf{k}}\widehat{L}^\dagger_{\mathbf{k}'} + \dots$$

and the states $|\Phi_T\rangle$ are those which do not contain any longitudinal or scalar photon:

$$a_{\mathbf{k},0}|\Phi_T\rangle = a_{\mathbf{k},3}|\Phi_T\rangle = 0$$



Symbolic picture of the Hilbert space of photons. In the shaded region, the Lorenz gauge condition is violated. States on the same fibers (thin lines) are gauge equivalent:[‡]

$$\langle\Phi|A_\mu(x)|\Phi\rangle = \langle\Phi_T|A_\mu(x)+\partial_\mu\Lambda(x)|\Phi_T\rangle = \langle\Phi_T|A_\mu(x)|\Phi_T\rangle + \partial_\mu\Lambda(x).$$

[‡] W.Greiner and J.Reinhardt, *Field Quantization*, Springer (1996).

't Hooft quantization for the EM field[§]

We define the following operators:

$$J_+ \equiv a^\dagger_{\mathbf{k},1} a_{\mathbf{k},2}\,, \quad J_- \equiv a^\dagger_{\mathbf{k},2} a_{\mathbf{k},1}\,, \quad J_3 \equiv \frac{1}{2}\left(a^\dagger_{\mathbf{k},1} a_{\mathbf{k},1} - a^\dagger_{\mathbf{k},2} a_{\mathbf{k},2}\right)$$

$$[J_+, J_-] = 2\,J_3\,, \quad [J_3, J_+] = +\,J_+\,, \quad [J_3, J_-] = -\,J_-$$

and

$$K_+ \equiv a^\dagger_{\mathbf{k},3} a_{\mathbf{k},0}\,, \quad K_- \equiv a^\dagger_{\mathbf{k},0} a_{\mathbf{k},3}\,, \quad K_3 \equiv \frac{1}{2}\left(a^\dagger_{\mathbf{k},0} a_{\mathbf{k},0} + a^\dagger_{\mathbf{k},3} a_{\mathbf{k},3}\right)$$

$$[K_+, K_-] = -2\,K_3\,, \quad [K_3, K_+] = +\,K_+\,, \quad [K_3, K_-] = -\,K_-$$

We have $su(2)$ algebra for the J operators and $su(1,1)$ algebra for the K operators.

Casimir operators:

$$J_0 = \frac{1}{2}\left(a^\dagger_{\mathbf{k},1} a_{\mathbf{k},1} + a^\dagger_{\mathbf{k},2} a_{\mathbf{k},2}\right)\,, \qquad K_0 = \frac{1}{2}\left(a^\dagger_{\mathbf{k},0} a_{\mathbf{k},0} - a^\dagger_{\mathbf{k},3} a_{\mathbf{k},3}\right).$$

[§]M.B., E.Celeghini, P.Jizba, F.Scardigli and G.Vitiello, arXiv:1801.06311 [quant-ph], to appear in the Proceedings of Symmetries in Science XVII.

Thus the Hamiltonian can be written as

$$H \;=\; \int d^3\mathbf{k}\,\omega_k\,(J_0 \,-\, K_0)$$

K_0 is responsible for the Hamiltonian to be not bounded from below
$\Rightarrow$ define the physical states as those for which

$$K_0|\psi\rangle_{phys} \,=\, 0$$

Such condition appears to be too restrictive, isolating only purely tranverse states. Thus we impose

$$_{phys}\langle\psi|K_0|\psi\rangle_{phys} \,=\, 0$$

which turns out to be equivalent to the Gupta–Bleuler condition.

Explicit form of the physical states:

$$|\psi\rangle_{phys} \;=\; \prod_{\mathbf{k}} |n_{\mathbf{k},1}\rangle_1 \otimes |n_{\mathbf{k},2}\rangle_2 \otimes |\alpha_{\mathbf{k}}\rangle$$

with $|\alpha_{\mathbf{k}}\rangle$ a generic state (to be determined) for the longitudinal and scalar photons.

We require:

$$\langle\alpha| \left(a^\dagger_{\mathbf{k},0} a_{\mathbf{k},0} - a^\dagger_{\mathbf{k},3} a_{\mathbf{k},3}\right) |\alpha\rangle \;=\; 0.$$

Furthermore, we restrict to states of the form $|\alpha\rangle = |\alpha\rangle_3 \otimes |\alpha\rangle_0$ where $|\alpha\rangle_3$ and $|\alpha\rangle_0$ denote (Glauber) coherent states for a_3 and a_0:

$$a_{\mathbf{k},3}|\alpha\rangle_3 \;=\; \alpha_k|\alpha\rangle_3$$

$$a_{\mathbf{k},0}|\alpha\rangle_0 \;=\; \alpha_k|\alpha\rangle_0$$

with the same α_k, for any $\mathbf{k}$.

The coherent state generators are

$$G_3(\alpha) = \exp \sum_{\mathbf{k}} \left(\alpha_k^* \, a_{\mathbf{k},3} - \alpha_k \, a_{\mathbf{k},3}^\dagger \right)$$

$$|\alpha\rangle_3 \; = \; G_3^{-1}(\alpha)|0\rangle$$

$$a_{\mathbf{k},3}(\alpha) \equiv G_3^{-1}(\alpha) a_{\mathbf{k},3} G_3(\alpha) \; = \; a_{\mathbf{k},3} - \alpha_k$$

and

$$G_0(\alpha) = \exp \sum_{\mathbf{k}} \left(-\alpha_k^* \, a_{\mathbf{k},0} + \alpha_k \, a_{\mathbf{k},0}^\dagger \right)$$

$$|\alpha\rangle_0 \; = \; G_0^{-1}(\alpha)|0\rangle$$

$$a_{\mathbf{k},0}(\alpha) G_0^{-1}(\alpha) a_{\mathbf{k},0} G_\alpha \; = \; a_{\mathbf{k},0} - \alpha_k \, .$$

The sign difference in the commutator for a_0 and $a_0^\dagger$ has dictated the choice of the sign in the definition of the G_0 generator. We thus obtain

$$\left(a_{\mathbf{k},0} - a_{\mathbf{k},3} \right) |\alpha\rangle \; = \; 0$$

$$\langle\alpha| \left(a_{\mathbf{k},0}^\dagger - a_{\mathbf{k},3}^\dagger \right) \; = \; 0 \, ,$$

which immediately extends to the physical states $|\psi\rangle_{phys}$.

Let us now consider the explicit form of the coherent states $|\alpha\rangle_0$ and $|\alpha\rangle_3$. We obtain

$$
\begin{aligned}
|\alpha\rangle_3 &= \exp\left(-\frac{1}{2}\int d^3k\,|\alpha_k|^2\right)\exp\left(\int d^3k\,\alpha_k\,a^\dagger_{\mathbf{k},3}\right)|0\rangle_3 \\
|\alpha\rangle_0 &= \exp\left(\frac{1}{2}\int d^3k\,|\alpha_k|^2\right)\exp\left(-\int d^3k\,\alpha_k\,a^\dagger_{\mathbf{k},0}\right)|0\rangle_0
\end{aligned}
$$

and

$$
\begin{aligned}
|\alpha\rangle &\equiv |\alpha\rangle_3 \otimes |\alpha\rangle_0 = \exp\left(\int d^3k\,\alpha_k\,\left(a^\dagger_{\mathbf{k},3} - a^\dagger_{\mathbf{k},0}\right)\right)|0\rangle \\
&= \left(1 + \int d^3k\,(-\alpha_k)\,L^\dagger_{\mathbf{k}} + \int d^3k d^3k'\,\frac{(-\alpha_k)(-\alpha_{k'})}{2!}L^\dagger_{\mathbf{k}}L^\dagger_{\mathbf{k}'} + \ldots\right)|0\rangle
\end{aligned}
$$

Thus a one-to-one correspondence exists between the coherent states above defined and those used in the Gupta-Bleuler quantization.

We can therefore identify the physical states of the Gupta-Bleuler condition with the ones defined by the 't Hooft condition.

Things to do...

– Find the beables for this system in terms of field components: to this end consider inversion formula for ladder operators

$$a_{\mathbf{k},\lambda} = i g_{\lambda,\lambda} \int d^3\mathbf{x} \frac{e^{ik\cdot x}}{\sqrt{2\omega_k (2\pi)^3}} \epsilon^\mu(\mathbf{k},\lambda) \Big(\dot{A}_\mu(x) - i\omega_k A_\mu(x) \Big)$$

where the polarization vectors satisfy the orthogonality relation:

$$\epsilon^\mu(\mathbf{k},\lambda)\epsilon_\mu(\mathbf{k},\lambda') = g_{\lambda,\lambda'}$$

– Construction of Hilbert space from above group structure $SU(2) \otimes SU(1,1)$.

– Emergence of gauge symmetry.

Thermo–field dynamics¶

Thermal averages ⇔ vacuum expectation values

$$\langle A \rangle = Z^{-1}(\beta) Tr \left[e^{-\beta H} A \right] = \langle 0(\beta)|A|0(\beta)\rangle$$

Needs to "double" the degrees of freedom:

$$|0(\beta)\rangle = Z^{-\frac{1}{2}}(\beta) \sum_n e^{-\frac{\beta}{2} E_n} |n, \tilde{n}\rangle$$

where $|n, \tilde{n}\rangle = |n\rangle \otimes |\tilde{n}\rangle$.

Thermal vacuum

$$|0(\theta)\rangle = \prod_{\mathbf{k}} \frac{1}{\cosh \theta_{\mathbf{k}}} \exp \left[\tanh \theta_{\mathbf{k}} a^\dagger_{\mathbf{k}} \tilde{a}^\dagger_{\mathbf{k}} \right] |0\rangle$$

¶Y.Takahashi and H.Umezawa, Collect. Phenom. (1975)

Number of particles in $|0(\theta)\rangle$

$$n_{\mathbf{k}} \equiv \langle 0(\theta)|a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}}|0(\theta)\rangle = \sinh^2\theta_{\mathbf{k}} = \frac{1}{e^{\beta\omega_{\mathbf{k}}} - 1}$$

gives the correct thermal average, i.e. the Bose–Einstein distribution.

"Thermal" Bogoliubov transformation:

$$a_{\mathbf{k}}(\theta) = a_{\mathbf{k}} \cosh\theta_{\mathbf{k}} - \tilde{a}^{\dagger}_{\mathbf{k}} \sinh\theta_{\mathbf{k}}$$

$$\tilde{a}_{\mathbf{k}}(\theta) = \tilde{a}_{\mathbf{k}} \cosh\theta_{\mathbf{k}} - a^{\dagger}_{\mathbf{k}} \sinh\theta_{\mathbf{k}}$$

where $\theta_{\mathbf{k}} = \theta_{\mathbf{k}}(\beta)$.

Thermal state condition:

$$\left(a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} - \tilde{a}^{\dagger}_{\mathbf{k}} \tilde{a}_{\mathbf{k}}\right) |0(\theta)\rangle \;=\; 0.$$

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Regular Article

A Physicist's view on Chopin's Études

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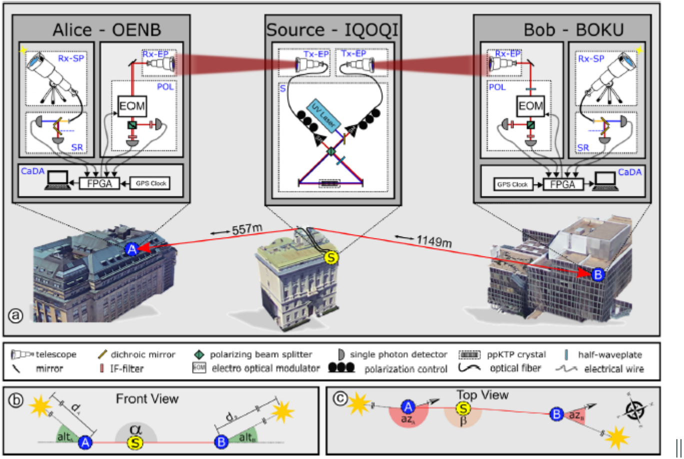
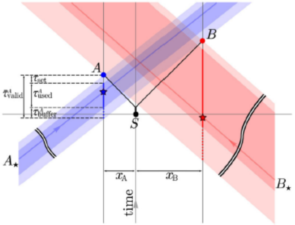
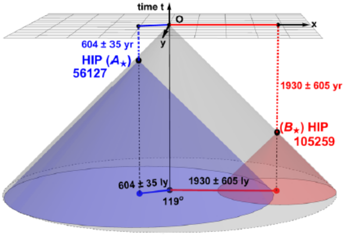
Abstract. We propose the use of specific dynamical processes and more in general of ideas from Physics to model the evolution in time of musical structures. We apply this approach to two Études by F. Chopin, namely Op.10 n.3 and Op.25 n.1, proposing some original description based on concepts of symmetry breaking/restoration and quantum coherence, which could be useful for interpretation. In this analysis, we take advantage of colored musical scores, obtained by implementing Scriabin's color code for sounds to musical notation.

Cosmic Bell Test



‖J.Handsteine et al., *Cosmic Bell Test: Measurement Settings From Milky Way Stars*, Phys. Rev. Lett. 118 060401 (2017)





Run	Side	HIP ID	az_k°	alt_k°	$d_k \pm \sigma_{d_k}$ [ly]	$\bar{\tau}_{valid}^k[\mu s]$	S_{exp}	p-value	ν
1	A	56127	199	37	604 ± 35	2.55	2.43	1.8×10^{-13}	7.3
	B	105259A	25	24	1930 ± 605	6.93			
2	A	80620	171	34	577 ± 40	2.58	2.50	4.0×10^{-33}	11.9
	B	2876	25	26	3624 ± 1370	6.85			