Localised States in Bounded Chiral Liquid Crystals

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Works

- L. Martina, M.V. Pavlov and S. Zykov: "Waves in the Skyrme Faddeev model and Integrable reductions", J. Phys. A: Math. Theor. **46** (2013) 275201
- L. Martina , M.V. Pavlov:"Magnetic domains and waves in the Skyrme Faddeev model", 2014 J. Phys.: Conf. Ser. **482** (2014) 012031
- G. De Matteis , L. Martina , V. Turco :"Skyrmion States In Chiral Liquid Crystals", to appear in Theoretical and Mathematical Physics, arXiv:1711.07922
- G. De Matteis, D. Delle Side, L. Martina, V. Turco: "Light Scattering by Cholesteric Skyrmions", arXiv:1802.07614, submitted to PRE
- G. De Matteis, L. Martina, V. Turco: " Helicoids in Chiral Liquid Crystals ", in preparation

Outline

Motivations

1. Localised objects in quantum/classical nonlinear theories
2. Topological charges
3. Skyrme-Faddeev model
   - Spin-Charge Separation of the pure Yang-Mills theory
4. Fundamental theories/ Condensed Matter Physics
   - $^{3}He - A$ superfluid
   - 2-band superconductor (Nb-doped $SrTiO_3$, MgB_2)
5. Planar Skyrmions in Magnetic Systems
   - Spin-Orbit Dzyaloshinskii-Moriya interaction
6. Application in information (quantum) technologies as storage/computation tools
7. Skyrmion/Spherulites in Liquid Crystals
   - Spontaneous chirality of the cholesteric phase
8. Defects in frustrated Chiral Liquid Crystals
   - Homeotropic anchoring boundary conditions
9. Integrability properties of the models

Basic References

- Faddeev, L.D.: Quantization of solitons. Princeton preprint IAS-75-QS70 (1975)
- Atiyah, M.: The Geometry and Physics of Knots. Cambridge Univ. P. (1990)
- Faddeev, L.D., Niemi, A.J.: Nature 387, 58 (1997)
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- Faddeev, L.D., Niemi, A.J.: Nucl. Phys. B 776, 38 (2007)
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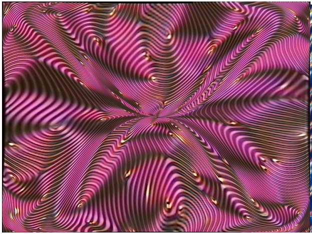
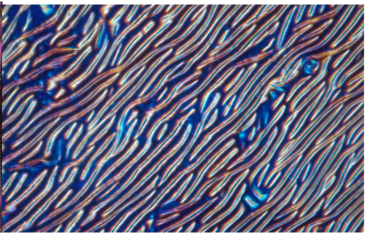
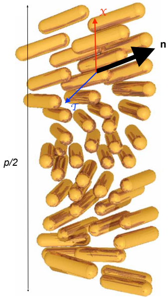
- C. Carboni, A. K. George, and A. Al-Lawati, Liquid Crystals (2004) 31
- I. Smalyukh , Y. Lansac, N.A. Clark, R. P. Trivedi, Nature Materials **9** (2010) 139
- A. O. Leonov *et al.*,Phys. Rev. E 90, 042502 (2014).
- Jun-ichi Fukuda, Sl. Žumer.Nature Communications 2 (2011) 246.
- P.J. Ackerman *et al.*, Phys. Rev. E **90** (2014).

- Stability of the order parameter configurations
- Topological ordering in disordered background
- Coexistence/Competition of short/long (UV/IR) wave modes
- Knotted and/or linked quasi-1-dimensional configurations
- Properties of knots and tangles

The Chiral Liquid Crystal Model



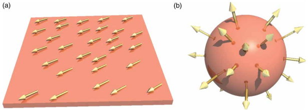
: Cholesteric ordering

$$\mathbf{n}(\mathbf{r}) \in \mathbb{RP}^2 \qquad \chi(\mathbf{r}) = \nabla \times \mathbf{n} \qquad \tau(\mathbf{r}) = \mathbf{n} \times \chi$$

$$\mathcal{E}_{FO} = \frac{K_1}{2}(\nabla \cdot \mathbf{n})^2 + \frac{K_2}{2}(\mathbf{n} \cdot \nabla \times \mathbf{n} - q_0)^2 + \frac{K_3}{2}(\mathbf{n} \times \nabla \times \mathbf{n})^2 + \frac{(K_2+K_4)}{2}\nabla \cdot [(\mathbf{n} \cdot \nabla)\mathbf{n} - (\nabla \cdot \mathbf{n})\mathbf{n}] - \frac{\varepsilon}{2}(\mathbf{n} \cdot \mathbf{E})^2$$

$$\mathcal{B} = \{(x, y, z) \in \mathbb{R}^3, \mid z \mid \leq \frac{L}{2}\}$$

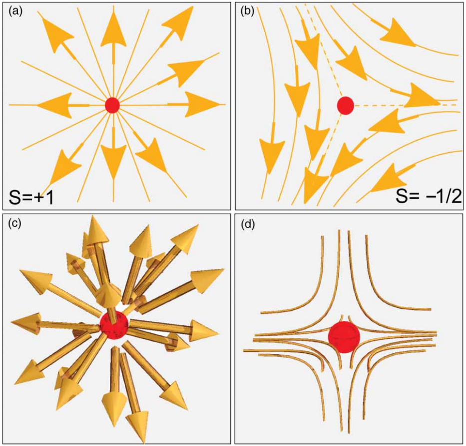
$\mathcal{E}_s = \frac{1}{2}K_s(1 + \alpha(\mathbf{n} \cdot \boldsymbol{\nu})^2)$ Rapini-Popoular anchoring



: planar homeotropic

$\Delta\Phi = \frac{\pi p \Delta\epsilon}{4\lambda^2\left(1-(\lambda/\lambda_0)^2\right)}$ de Vries optical rotation

Topological Point Defects in Nematics



: 2D textures with defects - winding charge: a) $S = 1$, b) $S = -1/2$. 2D textures with defect: a) $S = 1$, b) $S = -1/2$.
3D textures with defects: c) Hedgehog, d) Hyperbolic defect

Topological Line Defects in Cholesterics

$$\pi_1\left(SO\left(3\right)/D_2\right)=\{I,J,\imath,-\imath,\jmath,-\jmath,\ell,-\ell\}$$

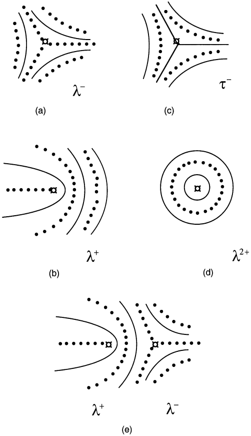


Figure 2
Wedge disclinations in a cholesteric: (a) λ^-, (b) $-\lambda^+$, (c) $-\tau^-$, (d) $-\lambda^{2+}$, and (e) $-\lambda^+$ and λ^- interacting (after Kleman and Lavrentovich 2000).

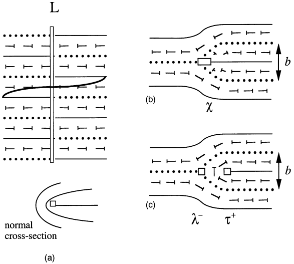
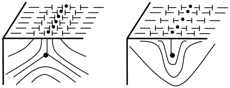
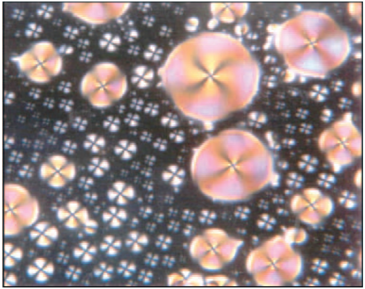


Figure 3
Equivalences: (a) χ^+ wedge disclination ≡ screw dislocation, χ twist disclination ≡ edge dislocation (after Kleman and Lavrentovich 2000).



s= - 1/2 s= + 1/2

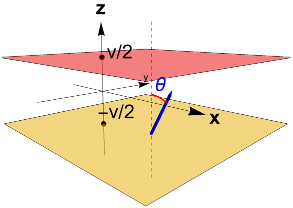
Spherulites



$$\mathbf{n}\left(\rho,\phi,z\right)=\cos\theta\left(\rho,z\right)\mathbf{k}+\sin\theta\left(\rho,z\right)\boldsymbol{\phi}\qquad\theta=\theta(\rho,z)$$

The equation

$$\frac{\partial^2\theta}{\partial z^2} + \frac{\partial^2\theta}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} - \frac{1}{\rho^2}\sin\theta\cos\theta \mp \frac{4\pi}{\rho}\sin^2\theta - \pi^4\left(\frac{E}{E_0}\right)^2\sin\theta\cos\theta = 0$$



$$\begin{cases} \theta(0,z) = \pi, \quad \theta(\infty,z) = 0, \\ \partial_z\theta\left(\rho, \pm\frac{\nu}{2}\right) = \mp 2\pi k_s \sin\theta\left(\rho, \pm\frac{\nu}{2}\right)\cos\theta\left(\rho, \pm\frac{\nu}{2}\right) \end{cases}$$

$p = \frac{2\pi}{|q_0|}$ $\qquad E_0 = \frac{\pi \mid q_0 \mid}{2}\sqrt{\frac{K}{\varepsilon}}$ $\qquad \nu = L/p$ $\qquad k_s = K_s/(Kq_0)$

Looking for solutions

$\rho \to 0$

Belavin-Polyakov

$$\theta = \arccos\left(\frac{\tilde{\rho}^2 - 4}{\tilde{\rho}^2 + 4}\right), \quad \tilde{\rho} = \frac{\rho}{\rho_0} \qquad \rho_0 = \frac{4}{\pi^3}\left(\frac{E_0}{E}\right)^2 = 4\pi\rho_1^2$$

conformal symmetry breaking

$$\theta(\rho, z) = \begin{cases} \pi - \frac{\rho}{\rho_0 Z(z)} & \rho/Z(z) < \pi\rho_0 \\ 0 & \rho/Z(z) > \pi\rho_0 \end{cases} \qquad Z(z) = 1 - \frac{2\pi k_s \cosh\left(\frac{z}{\rho_1}\right)}{2\pi k_s \cosh\left(\frac{\nu}{2\rho_1}\right) + \frac{1}{\rho_1}\sinh\left(\frac{\nu}{2\rho_1}\right)}$$

$\rho \to \infty$

$$\frac{\partial^2\theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} - \frac{1}{2\rho_1^2}\sin 2\theta = 0 \qquad \text{cylindrical Sine-Gordon equation}$$

$$\theta(\rho) = -i\,\ln\left(\frac{q(t)}{\sqrt{t}}\right), \qquad t = \left(\frac{\rho}{\rho_1}\right)^2$$

$$q'' = \frac{1}{q}q'^2 - \frac{1}{t}q' + \frac{q^3}{16t^2} - \frac{1}{16q},$$

the general Painlevé *III*

$$q'' = \frac{1}{q}q'^2 - \frac{1}{t}q' + \frac{q^2(a + c\,q)}{4t^2} + \frac{b}{4t} + \frac{d}{4q},$$

$a, b, c, d \in \mathbb{C}$

$$\rho \to \infty \qquad \theta \rightsquigarrow c_2\sqrt{\frac{\rho_1}{\rho}}\exp\left[-\frac{\rho}{\rho_1}\right].$$

The connection formulae

$$\rho \to 0: \qquad \theta\left(\rho|\alpha,\beta\right) \rightsquigarrow \alpha \ \ln\left(\frac{\rho}{\rho_1}\right) + i\frac{\pi}{2}\alpha + \beta + O\left(\left(\frac{\rho}{\rho_1}\right)^{2-|\Im\alpha|}\right), \qquad (|\Im\alpha| < 2)$$

and

$$\rho \to \infty \qquad \theta\left(\rho\right) \rightsquigarrow \left[b_+ e^{\frac{\rho}{\rho_1}}\left(\frac{\rho}{\rho_1}\right)^{-\frac{1}{2}+i\omega} + b_- e^{-\frac{\rho}{\rho_1}}\left(\frac{\rho}{\rho_1}\right)^{-\frac{1}{2}-i\omega}\right]\left(O\left(\frac{\rho_1}{\rho}\right)+1\right) + O\left(\left(\frac{\rho}{\rho_1}\right)\right.$$

$b_\pm$, ω related to the Cauchy data by the *connection formulas* (Novokshenov)

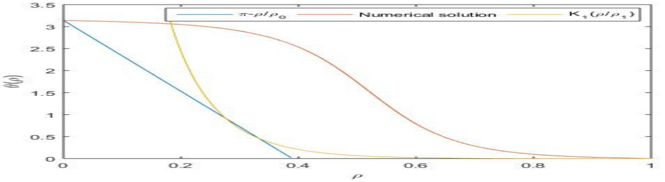
$$b_\pm = \frac{\mp(\pm)^{1/2} 2^{\pm 2i\omega} e^{-\pi\omega}}{\sqrt{\pi}} \Gamma(1 \mp i\omega) \frac{\sin(2\pi(\eta \pm \sigma))}{\sin(2\pi\eta)}, \ e^{-\pi\omega}\sin(2\pi\sigma) = \sin(2\pi\eta)$$

$$\sigma = \frac{1}{4} + \frac{i}{8}\alpha, \quad \eta = \frac{1}{4} + \frac{1}{4\pi}\left(\beta + \alpha \ \ln 8\right) + \frac{i}{2\pi} \ln \frac{\Gamma\left(\frac{1}{2} - \frac{i\,\alpha}{4}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\,\alpha}{4}\right)}. \quad b_- b_+ = -4i\omega, \quad |\Im\omega| < \frac{1}{2}$$

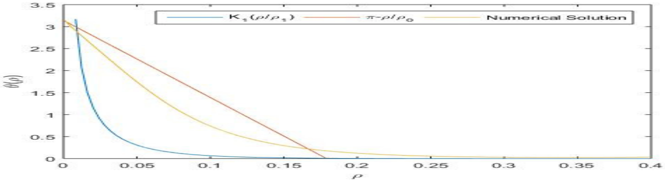
$$\omega = 0 \ b_+ = 0 \ b_- = -2i\sqrt{\frac{1}{\pi}}\cos(2\pi\sigma) \ \eta = -\sigma + \frac{1}{2} + k, \quad k \in \mathbb{Z}$$

$$\beta = -\left(\frac{i\pi}{2} + \ln 8\right)\alpha - 2i \ln \frac{\Gamma\left(\frac{1}{2} - \frac{i\,\alpha}{4}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\,\alpha}{4}\right)} + 4k\pi \qquad \alpha = -\frac{4}{\pi}\text{arcsinh}\left(\frac{\sqrt{\pi}}{2}c_2\right) \in \mathbb{R}^-$$

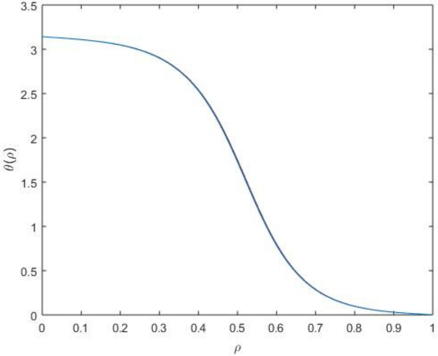
Analytical/ Numericals



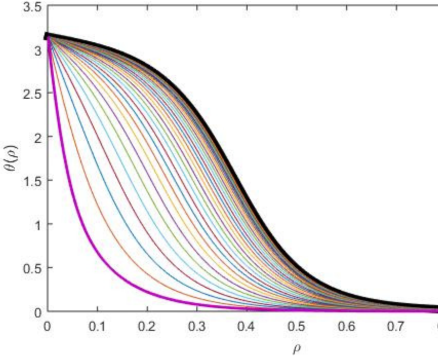
: Comparison between the numerical solution and the analytical linear approximations for $\frac{E}{E_0} = 1.02$.



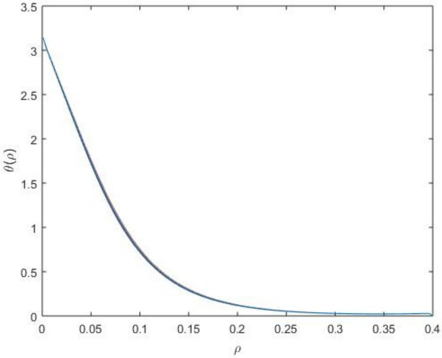
: Comparison between the numerical solution and the analytical linear approximations for $\frac{E}{E_0} = 1.5$



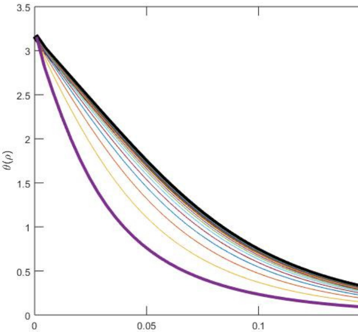
: $\left(\dfrac{E}{E_0}\right) = 1.02,\ k_s = 0.1,\ \nu = 1.8$



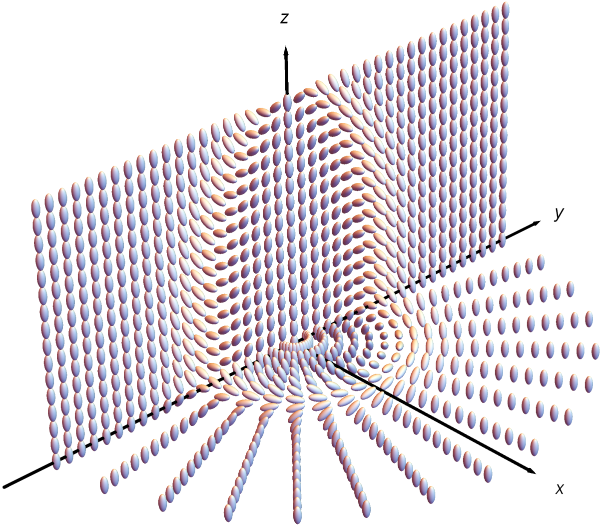
: $\left(\dfrac{E}{E_0}\right) = 1.02,\ k_s = 6,\ \nu = 1.8$



: $\left(\frac{E}{E_0}\right) = 1.5,\ k_s = 0.1,\ \nu = 1.8$



: $\left(\frac{E}{E_0}\right) = 1.5,\ k_s = 6,\ \nu = 1.8$



$$: \left(\frac{E}{E_0}\right) = 1.02,\ k_s = 6,\ \nu = 1.8$$

$$\nabla\left(\nabla\cdot\boldsymbol{E}\right)-\nabla^{2}\boldsymbol{E}=-\partial_{tt}\boldsymbol{D}$$
$$\boldsymbol{D}=\epsilon_{\perp}\boldsymbol{E}+\Delta\epsilon\,\mathbf{n}\left(\boldsymbol{E}\cdot\mathbf{n}\right),\qquad\Delta\epsilon=\epsilon_{\parallel}-\epsilon_{\perp}$$

1. the liquid crystal molecules are not deformed/rotated by wave
2. $\lambda\approx\rho_0$, i.e. $k\left(\omega\right)\gtrsim\frac{1}{\rho_0}$
3. negligible effects of the bounding surfaces
4. $\nabla\cdot\boldsymbol{E}\cong 0$
5. $\theta=\theta\left(\rho\right)$

$$\boldsymbol{\mathcal{E}}=\mathcal{E}_{\rho}\left(\rho,\phi,z\right)\,\boldsymbol{\rho}+\mathcal{E}_{\phi}\left(\rho,\phi,z\right)\,\boldsymbol{\phi}+\mathcal{E}_{z}\left(\rho,\phi,z\right)\,\mathbf{k}$$

$$\nabla^{2}\boldsymbol{\mathcal{E}}=-k^{2}\mathcal{Q}\,\boldsymbol{\mathcal{E}},\ \mathcal{Q}=\mathbf{1}_{3}+\frac{\Delta\epsilon}{\epsilon_{\perp}}\mathbf{n}\otimes\mathbf{n},\qquad k=\frac{\omega}{c}\sqrt{\epsilon_{\perp}}\quad\tilde{k}=k\sqrt{1+\frac{\Delta\epsilon}{\epsilon_{\perp}}}$$

$$\left(\nabla^{2}+k^{2}\right)\left(\mathcal{E}_{\rho}\,\boldsymbol{\rho}+\mathcal{E}_{\phi}\,\boldsymbol{\phi}+\mathcal{E}_{z}\,\mathbf{k}\right)=$$
$$-k^{2}\frac{\Delta\epsilon}{\epsilon_{\perp}}\left[\left(\sin\theta\,\mathcal{E}_{\phi}+\cos\theta\,\mathcal{E}_{z}\right)\cos\theta\,\mathbf{k}+\left(\sin\theta\,\mathcal{E}_{\phi}+\cos\theta\,\mathcal{E}_{z}\right)\sin\theta\,\boldsymbol{\phi}\right].$$

$$\mathcal{E}_{\rho}\rightsquigarrow\mathcal{E}_{\infty\rho}\sin\phi\,e^{\imath k\rho\cos\phi},\ \mathcal{E}_{\phi}\rightsquigarrow\mathcal{E}_{\infty\phi}\cos\phi\,e^{\imath k\rho\cos\phi},\ \mathcal{E}_{z}\rightsquigarrow\mathcal{E}_{\infty z}e^{\imath\tilde{k}\rho\cos\phi}\quad\phi\to\pm\pi\quad\rho\to\infty.$$

The *out plane* conversion

$$\mathcal{E}_z(\boldsymbol{r}) = \mathcal{E}_{\infty z} e^{\imath \tilde{k} \rho \cos\phi} + \int G(\boldsymbol{r}, \boldsymbol{r}') \, U\left[\mathcal{E}_z(\boldsymbol{r}'), \mathcal{E}_\phi(\boldsymbol{r}'), \theta(\rho')\right] \, d\boldsymbol{r}'$$

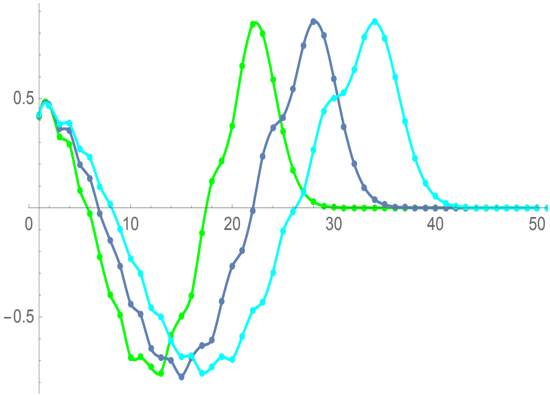
$$U\left[\mathcal{E}_z(\boldsymbol{r}), \mathcal{E}_\phi(\boldsymbol{r}), \theta(\rho)\right] = -k^2 \frac{\Delta\epsilon}{\epsilon_\perp} \left(\frac{1}{2} \sin 2\theta(\rho) \, \mathcal{E}_\phi(\boldsymbol{r}) - \sin^2\theta(\rho) \, \mathcal{E}_z(\boldsymbol{r}) \right)$$

$$\mathcal{E}_z^B(\boldsymbol{r}) = (1 - \imath) \, \mathcal{E}_{\infty\phi} \frac{\pi \, \Delta\epsilon}{8 \, \epsilon_\perp} \frac{e^{\imath \, \tilde{k} \, \rho}}{\sqrt{\pi \, \tilde{k} \, \rho}} \left[\mathcal{I}_0^\phi + 2 \sum_{m=1}^{+\infty} \mathcal{I}_m^\phi \cos m\phi \right],$$

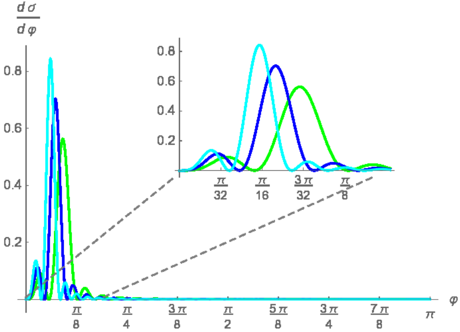
$$\frac{d\,\sigma_{conv}}{d\phi}(\hat{r}, \hat{z}; \hat{x}, \hat{y}) = \frac{\pi}{32} \sqrt{\frac{\epsilon_\perp}{\epsilon_{||}}} \left(\frac{\Delta\epsilon}{\epsilon_\perp} \right)^2 \frac{\nu\,\rho_0}{k\,\rho_0} \left[\mathcal{I}_0^\phi + 2 \sum_{m=1}^{+\infty} \mathcal{I}_m^\phi \cos m\phi \right]^2,$$

$$\sigma_{conv} = \frac{\pi^2}{16} \sqrt{\frac{\epsilon_\perp}{\epsilon_{||}}} \left(\frac{\Delta\epsilon}{\epsilon_\perp} \right)^2 \frac{\nu\,\rho_0}{k\rho_0} \left[\left(\mathcal{I}_0^\phi \right)^2 + 2 \sum_{m=1}^{\infty} \left(\mathcal{I}_m^\phi \right)^2 \right],$$

$$\mathcal{I}_m^\phi(k\rho_0) = -\int_0^{\pi k \rho_{\mathbf{0}}} \sin\left(2 \frac{s}{k\rho_0} \right) \left(J_m(s)^2 \right)' \, s \, ds,$$



: The numerical values of $\mathcal{I}_m^{\phi}$ as function of $0 \leq m \leq 50$ for three different values of $k\rho_0$, precisely 8 (green), 10 (blue) and 12 (cyan).



: The numerical evaluation of the conversion cross section (21), in arbitrary units, for the three different values of $\tilde{k}\rho_0 = 8, 10, 12$.

The *out plane* conversion

$$\begin{pmatrix} L+k^2 & -M \\ M & L+k^2 \end{pmatrix} \begin{pmatrix} \mathcal{E}_\rho \\ \mathcal{E}_\phi \end{pmatrix} = k^2 \frac{\Delta\epsilon}{\epsilon_\perp} \begin{pmatrix} 0 \\ \sin^2\theta\, \mathcal{E}_\phi + \frac{1}{2}\sin 2\theta\, \mathcal{E}_z \end{pmatrix},$$

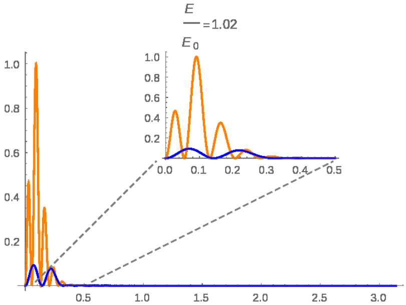
$$L = \nabla_0^2 - \tfrac{1}{\rho^2}, \quad M = \tfrac{2}{\rho^2}\,\partial_\phi \;\; \mathcal{E}_{\infty\rho} = \mathcal{E}_{\infty\phi} = 0, \mathcal{E}_{\infty z} \neq 0$$

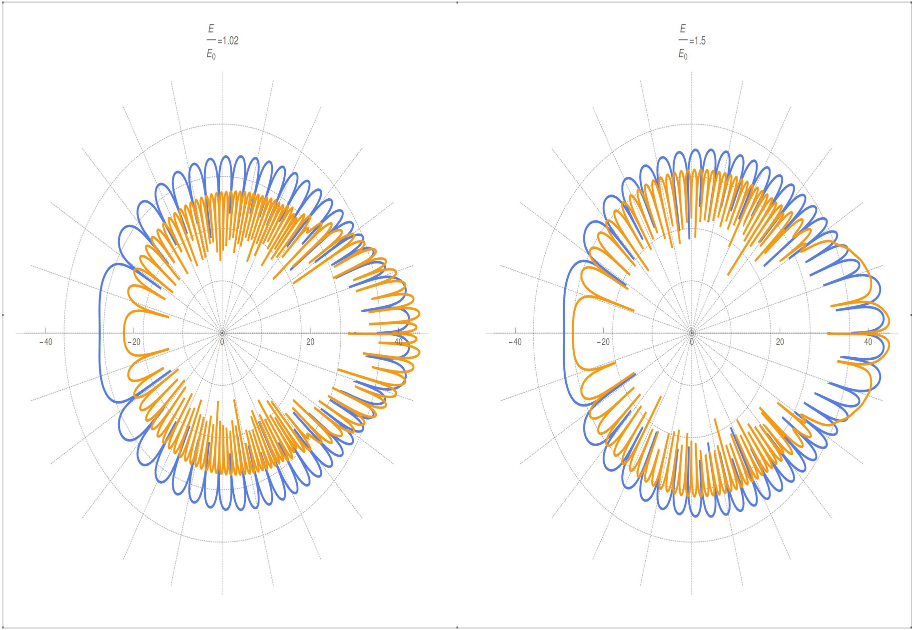
$$\begin{pmatrix} \mathcal{E}_\rho^B \\ \mathcal{E}_\phi^B \end{pmatrix} = \frac{\mathcal{E}_{\infty z}\sqrt{\pi}\,\Delta\epsilon}{2^{\frac{5}{2}}\,\epsilon_\perp} e^{-\imath\,\frac{\pi}{4}} \frac{e^{\imath\,k\,\rho}}{\sqrt{k\,\rho}} \begin{pmatrix} 2\sum_{m=1}^{\infty} I_m^{(\rho)} \sin m\phi \\ I_0^{(\phi)} + 2\sum_{m=1}^{\infty} I_m^{(\phi)} \cos m\phi \end{pmatrix}.$$

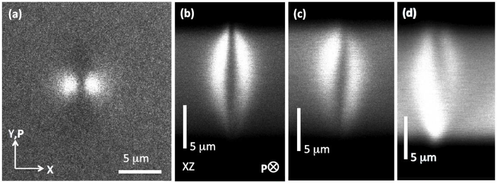
$$I_m^{(\rho)} = \frac{\tilde{k}^2}{k^2}\int \sin 2\theta \left(\frac{s}{\tilde{k}}\right) J_m(s) \left[J_{m-1}\left(k\,\frac{s}{\tilde{k}}\right) + J_{m+1}\left(k\,\frac{s}{\tilde{k}}\right)\right] s\,ds\,d\phi'$$

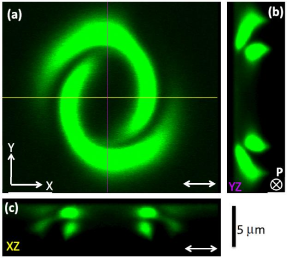
$$I_m^{(\phi)} = \frac{\tilde{k}^2}{k^2}\int \sin 2\theta \left(\frac{s}{\tilde{k}}\right) J_m(s) \left[J_{m-1}\left(k\,\frac{s}{\tilde{k}}\right) - J_{m+1}\left(k\,\frac{s}{\tilde{k}}\right)\right] s\,ds,$$

$$\frac{d\,\sigma}{d\phi} = \frac{\pi}{32}\sqrt{\frac{\epsilon_\perp}{\epsilon_{||}}}\left(\frac{\Delta\epsilon}{\epsilon_\perp}\right)^2 \frac{\nu\,\rho_0}{k\,\rho_0} \left[4\left(\sum_{m=1}^{\infty} I_m^{(\rho)} \sin m\phi\right)^2 + \left(I_0^{(\phi)} + 2\sum_{m=1}^{+\infty} I_m^{(\phi)}\cos m\phi\right)^2\right]$$




$\frac{E}{E_0}$=1.02
$\frac{E}{E_0}$=1.5
-40
-20
0
20
40





arXiv:1612.09015 P.J. Ackerman et al.

2π-Helicoids

$$\mathbf{n}\left(x,y,z=\pm\frac{L}{2}\right)=\mathbf{k}\rightarrow\mathbf{n}\left(\mathbf{r}\right)=\left(0,-\sin\theta\left(x,\ z\right),\cos\theta\left(x,\ z\right)\right)$$

$$\partial_x^2\theta+\partial_z^2\theta=\frac{\Lambda^2}{2}\sin 2\theta,\quad \theta\left(x,\ \pm\frac{L}{2}\right)=k\pi\quad\left(\Lambda=\frac{\pi q_0 E}{2E_0}\right)$$

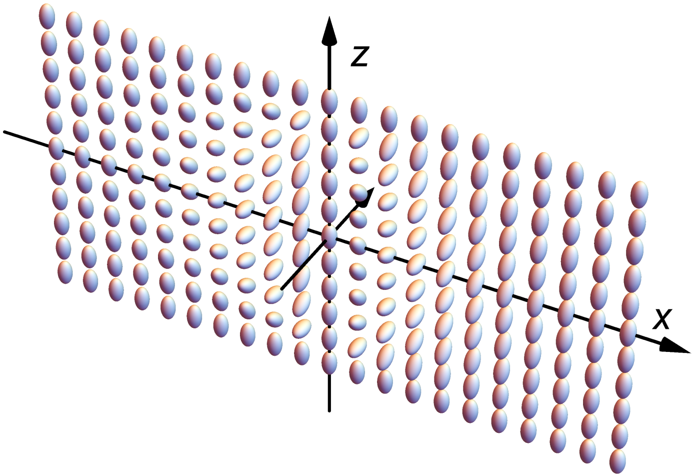
$$\theta=2\arctan\left[X\left(x\right)Z\left(z\right)\right]$$

$$X^{''}=2aX^3+\left(\Lambda^2-d\right)\ X,\qquad \frac{Z^{''}}{Z}=-d-\frac{2a}{Z^2}+2\left(\frac{Z'}{Z}\right)^2,$$

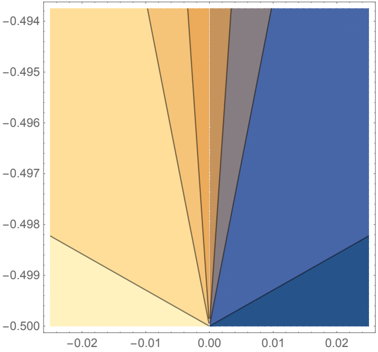
$$\Lambda^2-d>0\qquad X\left(x\right)=\pm\sqrt{\frac{\Lambda^2-d}{a}}\operatorname{csch}\left(\sqrt{\Lambda^2-d}\,x\right),$$

$$\begin{aligned}\theta_n\left(x,z\right)&=2\tan^{-1}\left[\frac{c_n\,\ell}{\pi\left(1+2n\right)}\cos\left(\frac{\pi\left(1+2n\right)z}{L}\right)\operatorname{csch}\left(c_n\Lambda\,x\right)\right]-\operatorname{sign}\left(x\right)\pi\\&=2\operatorname{sign}\left(n\right)\cot^{-1}\left[\frac{\pi\left(1+2n\right)}{c_n\,\ell}\sec\left(\frac{\pi\left(1+2n\right)z}{L}\right)\sinh\left(c_n\Lambda\,x\right)\right]+\operatorname{sign}\left(x\right)\pi,\quad n\in\mathbb{Z},\end{aligned}$$

$$\ell=\Lambda\,L,\qquad c_n=\left[1+\frac{(1+2n)^2\pi^2}{\Lambda^2L^2}\right]^{\frac{1}{2}}.$$


z
x

Disclinations



$$\theta_0(x,z) = -\pi + 2t - \frac{\rho^2 \sin(2t)\left(\Lambda^2 L^2 \cos^2 t + \pi^2\right)}{6L^2} + O\left(\left(\frac{\rho}{L}\right)^4\right) 0 \le t \le \pi,$$

Energetics of the Helicoids

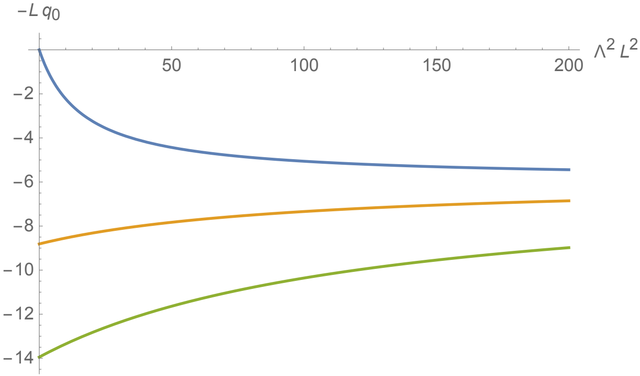
$\mathcal{E}_{max} \approx \frac{K}{4a^2}$
$\left(aL \leq \rho \leq \frac{L}{2}\right) \times (0 \leq t \leq \pi)$

$$\frac{\Delta E^a}{2K} = -4\pi \log(2a) - \frac{Lq_0}{180}\left[5\left(72+\pi^2\right) + \left(3-24a^3\right)\Lambda^2 L^2 - 720a - 40\pi^2 a^3\right]$$

$$\frac{\Delta E_{Hel}}{2K} \approx L\left[q_H - q_0\left(\frac{\pi^2}{36} + 2 - 2\sqrt{\frac{K}{\mathcal{E}_{max}} + \frac{\Lambda^2 L^2}{60}}\right)\right],.$$

$q_H \simeq \frac{2\pi}{L}\log\left(\frac{\mathcal{E}_{max}}{K}\right) + \frac{\pi}{32}\Lambda^2 L$

$$q_0^{tr} = \frac{45\pi\left(64\log\left(\frac{\mathcal{E}_{max}}{K}\right) + \Lambda^2 L^2\right)}{8L\left[3\Lambda^2 L^2 + 5\pi^2 - 360\left(\sqrt{\frac{K}{\mathcal{E}_{max}}} - 1\right)\right]}.$$


$-L\,q_0$
$\Lambda^2 L^2$
50
100
150
200
−2
−4
−6
−8
−10
−12
−14

π-Helicoids

$$\theta\left(x, z = \pm\frac{L}{2}\right) = \begin{cases} \pi & x < 0 \\ 0 & x > 0 \end{cases}$$

$$\theta\,(x, z) \to \theta\,(x, z) = \theta\,(\Lambda x, \Lambda z)\,, \quad \ell = \Lambda\, L,$$

$$\partial_x^2\theta + \partial_z^2\theta = \tfrac{1}{2}\sin 2\theta, \qquad \theta\left(x, z = \pm\frac{\ell}{2}\right) = \begin{cases} \pi & x < 0 \\ 0 & x > 0 \end{cases}$$

Linear π-Helicoid

$|\theta| \ll 1$

$$\partial_x^2\theta_+ + \partial_z^2\theta_+ = \theta_+, \qquad \begin{array}{cc} \theta_+\left(x, z = \pm\ell/2\right) = 0 & \forall\ x > 0 \\ \theta_+\left(x = 0^+, z\right) = \frac{\pi}{2} & \forall\ |z| < \frac{\ell}{2} \end{array}$$

$$\theta_-\left(x, z\right) = \pi - \theta_+\left(-x, z\right)$$

$$\theta_+\left(x, z\right) = -\frac{1}{\pi}\left[\int_{-\infty}^{0} e^{\Omega(\lambda)x + \omega(\lambda)\left(z + \frac{\ell}{2}\right)} G_1\left(\lambda\right)\frac{d\lambda}{\lambda} + \int_{\infty}^{0} e^{\Omega(\lambda)x + \omega(\lambda)\left(z - \frac{\ell}{2}\right)} G_1\left(\lambda\right)\frac{d\lambda}{\lambda}\right.$$

$$\left. - \int_{-\imath\infty}^{0} e^{\Omega(\lambda)x + \omega(\lambda)\left(z + \frac{\ell}{2}\right)} G_2\left(\lambda\right)\frac{d\lambda}{\lambda}\right],$$

Y.Antipov and A.S. Fokas , *Math. Proc. Camb. Phil Soc.* **138**, 339-365 (2005).

$\lambda = \mathbb{C}_\lambda / \{0\}$

$$\Omega\left(\lambda\right) = \frac{\imath}{2}\left(\frac{1}{\lambda} - \lambda\right), \qquad \omega\left(\lambda\right) = \frac{1}{2}\left(\frac{1}{\lambda} + \lambda\right)$$

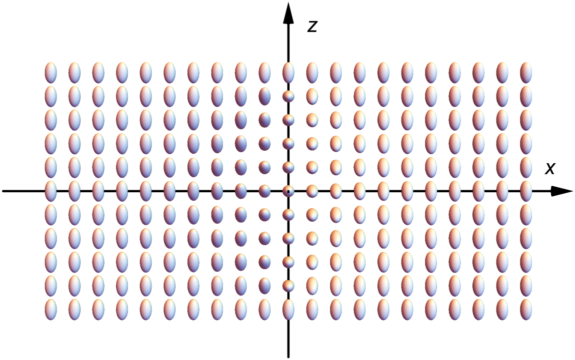
$$G_1\left(\lambda\right) = \frac{\imath\pi}{4}\frac{1-\lambda^2}{1+\lambda^2}\frac{e^{\omega(\lambda)\ell}-1}{e^{\omega(\lambda)\ell}+1}, \qquad G_2\left(\lambda\right) = \frac{\imath\pi}{4}\frac{1-\lambda^2}{1+\lambda^2}\left(1 - e^{-\omega(\lambda)\ell}\right)$$

$$P_G = \left\{ -\frac{i\left(\sqrt{\ell^2+(2n+1)^2\pi^2} - \pi(2n+1)\right)}{\ell} \right\}_{n\in\mathbb{Z}}$$

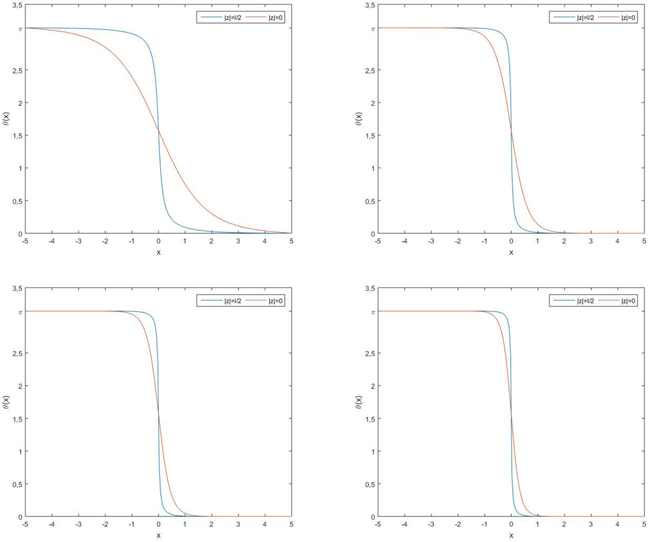
$$\theta_+\left(x,z\right) = 2\sum_{k=0}^{+\infty} \frac{\left(-1\right)^k}{2k+1} e^{-\frac{x\sqrt{\pi^2(2k+1)^2+\ell^2}}{\ell}} \cos\left(\frac{\pi(2k+1)z}{\ell}\right).$$

34

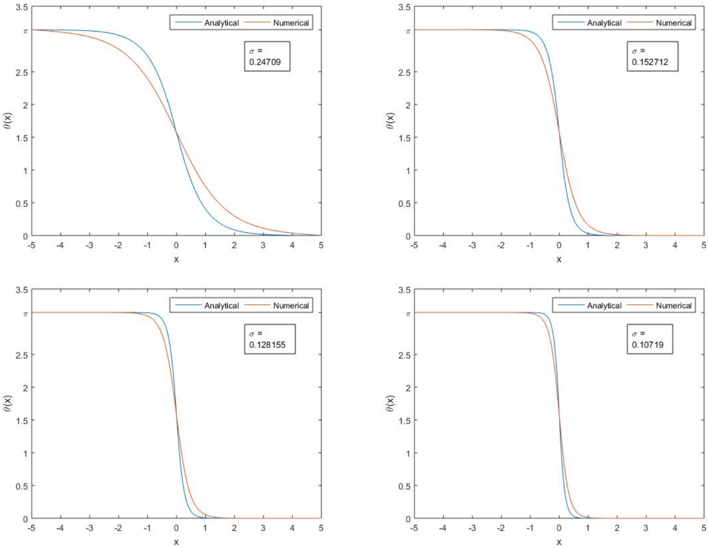
$\Delta x \approx \frac{2\ell}{\sqrt{\ell^2+\pi^2}}$,



: π-helicoid with strong homeotropic boundary conditions. It has been used the formula (35) up to $k = 10$. Cross section at $y = 0$. The ellipsoids at $\left(0, z = \pm\frac{\ell}{2}\right)$ overlaps with ellipsoids lying along the y axis: disclinations uniformly extended in y



: (a) $\Lambda^2 = 0.5$, (b) $\Lambda^2 = 3.5$, (c) $\Lambda^2 = 7$, (d) $\Lambda^2 = 14$.



: Numerical / linear approximation (35) at $z = 0$. (a) $\Lambda^2 = 0.5$, (b) $\Lambda^2 = 3.5$, (c) $\Lambda^2 = 7$, (d) $\Lambda^2 = 14$.

NonLinear π-Helicoids

$\Theta_+ = 2\theta$

$$\partial_x^2\Theta + \partial_z^2\Theta = \sin\Theta, \qquad \begin{cases} \Theta\left(x, z = \pm\frac{\ell}{2}\right) = 0 & \forall\, x > 0 \\ \Theta\left(x = 0^+, z\right) = \pi & \forall\, |z| < \frac{\ell}{2} \end{cases} .$$

$$\partial_x\Phi + \frac{\Omega(\lambda)}{2}[\sigma_3, \Phi] = V_1(x, z, \lambda)\,\Phi$$
$$\partial_y\Phi + \frac{\omega(\lambda)}{2}[\sigma_3, \Phi] = V_2(x, z, \lambda)\,\Phi,$$

$$V_1(x, z, \lambda) = \frac{\imath}{4\lambda}\Big\{\lambda\frac{\partial_x\Theta}{4}\sigma_1 + \imath\left[\sinh 2\kappa\,\lambda^2 - \sinh(2\kappa - \imath\Theta)\right]\sigma_2 + \left[(\cosh 2\kappa - 1)\,\lambda^2 - \cosh(2\kappa - \imath\Theta) + 1\right]\sigma_3\Big\},$$

Fokas, A. S. and Lenells, J. and Pelloni, B, Boundary Value Problems for the Elliptic Sine-Gordon Equation in a Semi-strip, *J. Nonlin. Sci.* **23** (2013), 241–282

Conclusions

1. Chiral Liquid Crystals are described by models which have properties similar with other non linear classical field theories.
2. They have 1D extended disclination singularity.
3. CLC, frustrated by adding proper boundary conditions, possess 2D skyrmionic solutions (spherulites). They are stabilised by a topological charge. They may have point singularities at the boundary surfaces.
4. The equation for the spherulite profile is not integrable, but posses bounded solutions, which cannot be obtained by perturbing the PIII equation.
5. Actually spherulite lives in 3D. The corresponding optical properties, in particular polarisation conversion is studied in its equatorial plane. They are compatible with the experimental observations.

6. Helicoid solutions, cal be studied in terms of deformed kinks of the sine-Gordon equation with boundaries.
7. The 2π-Helicoids possess linear disclination singularities on the boundary surfaces.
8. The π-Helicoids are asymptotically described by a linear theory.
9. The exact π-Helicoid is a difficult boundary value problems, partially by Fokas *et al*.
10. The full solution of such a problem will be our future aim.