Derivation based differential calculi for noncommutative algebras deforming a class of three dimensional spaces

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- The aim of this talk is to present the setting and the main results of a paper written with G.Marmo and P.Vitale (2018).

- It concerns the problem of introducing a differential calculus on a family of algebras A with a 3d Lie type non commutativity.

- The algebras A will be realised as subalgebras of $\mathcal{M}^\theta$, the Moyal 4d algebra. The differential calculi will be constructed starting from a suitable Lie algebra of (undeformed) derivations for $\mathcal{M}^\theta$ which can be reduced to A.

What is a derivation based differential calculus?

- On an orientable N-dim. differentiable manifold M, the differential calculus is the differential graded algebra

$$(\Omega(M) = \oplus_{k=0}^{N} \Omega_k(M), \wedge, \mathrm{d}, \mathrm{d}^2 = 0),$$

with $\mathcal{F}(M) = \Omega_0(M)$.

- The $\mathcal{F}(M)$-bimodule $\Omega_1(M)$ is dual to the module $\mathfrak{X}(M)$ of vector fields, which coincides with the space of all derivations for $\mathcal{F}(M)$.

$$i_X : \Omega_k(M) \to \Omega_{k-1}(M), \qquad L_X = \mathrm{d}i_X + i_X\mathrm{d}$$

- The set $\mathcal{X}(M)$ is an infinite dimensional Lie algebra w.r.t.the commutator

$$[X_1, X_2]f = X_1(X_2 f) - X_2(X_1 f)$$

- When, following the Gelfand duality approach, $\mathcal{F}(M)$ is replaced by a (non commutative) algebra $\mathcal{A}$, the problem of defining a differential calculus for it has been widely studied (see the spectral triple formalism for C^*-algebras, the covariant calculus approach for quantum spaces and groups, the twisted-deformed approach)

- We start from a (finite dimensional) Lie algebra of derivations, i.e. $\rho : \mathfrak{l} \to \mathrm{End}(\mathcal{A})$ with

$$[\rho(X_a), \rho(X_b)] = \rho([X_a, X_b])$$
$$\rho(X)(a_1 a_2) \;=\; (\rho(X)a_1)a_2 + a_1(\rho(X)a_2)$$

- We denote by $C^n_\wedge(\mathfrak{l}, \mathcal{A})$ the set of $Z(\mathcal{A})$-multilinear alternating mappings

$$\omega \;:\; X_1 \wedge \cdots \wedge X_n \;\mapsto\; \omega(X_1, \ldots, X_n)$$

from $\mathfrak{l}^{\otimes n}$ to $\mathcal{A}$

- On the graded vector space $C_\wedge(\mathfrak{l},\mathcal{A}) = \oplus_{j=0}^{j=\dim\mathfrak{l}} C^n_\wedge(\mathfrak{l},\mathcal{A})$, with $C^0_\wedge(\mathfrak{l},\mathcal{A}) = \mathcal{A}$, one can define
  - a wedge product

$$(\omega \wedge \omega')(X_1, \ldots, X_{k+s}) = \frac{1}{k!s!} \sum_{\sigma \in \mathcal{S}_{k+s}} (\mathrm{sign}(\sigma))\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\omega'(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+s)})$$

  - an operator $\mathrm{d} : C^n_\wedge(\mathfrak{l},\mathcal{A}) \to C^{n+1}_\wedge(\mathfrak{l},\mathcal{A})$ by

$$(\mathrm{d}\omega)(X_0, X_1, \ldots, X_n) = \sum_{k=0}^{n} (-1)^k \rho(X_k)(\omega(X_0, \ldots, \hat{X}_k, \ldots, X_n)) + \frac{1}{2}\sum_{r,s} (-1)^{k+s}\omega([X_r, X_s], X_0, \ldots, \hat{X}_r, \ldots, \hat{X}_s, \ldots, X_n)$$

  (with $\hat{X}_r$ denoting that the r-th term is omitted), such that d a graded antiderivation with $\mathrm{d}^2 = 0$, so $(C_\wedge(\mathfrak{l},\mathcal{A}), \wedge, \mathrm{d})$ is a graded differential algebra.

- On 1-forms we have an $\mathcal{A}$-bimodule structure with

$$(f_1 \mathrm{d} f_2)(X) \;=\; f_1 \left(\rho(X) f_2\right) \qquad \text{and} \qquad ((\mathrm{d} f_2) f_1)(X) \;=\; (\rho(X) f_2) f_1$$

- The operator

$$(i_X \omega)(X_1, \ldots, X_n) \;=\; \omega(X, X_1, \ldots, X_n)$$

gives a degree (-1) antiderivation from $C_\wedge^{n+1}(\mathfrak{l}, \mathcal{A}) \to C_\wedge^n(\mathfrak{l}, \mathcal{A})$.

- The operator defined by $L_X = i_X \mathrm{d} + \mathrm{d} i_X$ is the degree zero Lie derivative along X, so we have an exterior Cartan calculus on $\mathcal{A}$ depending on a Lie algebra of derivations.

- This exterior algebra is an example of a derivation based calculus.
  We denote it by $\underline{\Omega}_{\mathfrak{l}}(\mathcal{A})$.

- The subset $\Omega_{\mathfrak{l}}(\mathcal{A}) \subset \underline{\Omega}_{\mathfrak{l}}(\mathcal{A})$ is defined as the smallest differential graded subalgebra of $\underline{\Omega}_{\mathfrak{l}}(\mathcal{A})$ generated in degree 0 by $\mathcal{A}$.
  By construction, every element in $\Omega^n_{\mathfrak{l}}(\mathcal{A})$ can be written as $a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n$ terms with $a_j \in \mathcal{A}$, while this is not necessary for elements in $\underline{\Omega}_{\mathfrak{l}}(\mathcal{A})$.

- In terms of dual modules, it is

$$(\Omega^1_{\mathfrak{l}}(\mathcal{A}))^* = \mathrm{Der}_{\mathfrak{l}}(A) \qquad \text{and} \qquad (\mathrm{Der}_{\mathfrak{l}}(A))^* = \underline{\Omega}_{\mathfrak{l}}(\mathcal{A})$$

- This difference will be seen in some of the examples we shall describe.
  If $\mathcal{A} = \mathcal{F}(M)$ for a paracompact manifold, then $\underline{\Omega}_{\mathfrak{l}}(\mathcal{A}) = \Omega_{\mathfrak{l}}(\mathcal{A})$.

- Given $(\mathbb{R}^{2N}, \omega = \mathrm{d}q_a \wedge \mathrm{d}p_a)$, for $f, g \in \mathcal{S}(\mathbb{R}^{2N})$ one defines the Moyal product via (with $\theta > 0$)

$$(f * g)(x) = \frac{1}{(\pi\theta)^{2N}} \int\int \mathrm{d}u\mathrm{d}v\, f(x+u)g(x+v)\, e^{-2i\omega^{-1}(u,v)/\theta},$$

The set $\mathcal{A}_\theta = (\mathcal{S}(\mathbb{R}^{2N}), *)$ is a non unital pre C^*-algebra.

- The set $\mathcal{M}^\theta = \mathcal{M}_L^\theta \cap \mathcal{M}_R^\theta$ of multipliers is a unital $*$-algebra, and provides the maximal compactification of $\mathcal{A}_\theta$ defined by duality. It contains polynomial, plane waves, Dirac's δ and its derivatives.
The set $(\mathcal{M}^\theta, *)$ is what we call the Moyal algebra.

- The Moyal product is a non commutative deformation of the pointwise one

$$f * g \sim fg + \frac{i\theta}{2}\{f, g\} + \sum_{k=2}^{\infty} (\frac{i\theta}{2})^k \frac{1}{k!} D_k(f, g) \qquad \text{as } \theta \to 0$$

- The commutator deforms the Poisson structure

$$[f,g]_\theta = f*g - g*f = i\theta\,\{f,g\} + \sum_{s=1}^{\infty} \frac{2}{(2s+1)!}\left(\frac{i\theta}{2}\right)^{2s+1} D_{2s+1}(f,g).$$

- For degree 1 polynomials we have the CCR

$$[q_a,q_b]_\theta = 0, \qquad [p_a,p_b]_\theta = 0, \qquad [q_a,p_b]_\theta = i\theta\delta_{ab}$$

while, if $f,g \in S = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \mathcal{P}_2$,

$$[f,g]_\theta = i\theta\{f,g\}.$$

- So $(S,\{\ ,\ \})$ is a Poisson subalgebra of $\mathcal{F}(\mathbb{R}^4)$, while $(S,[\ ,\ ]_\theta)$ is a Lie subalgebra in $\mathcal{M}^\theta$ w.r.t. the $*$-product commutator.
It is isomorphic to a one dimensional central extension of the Lie algebra $\mathfrak{isp}(4,\mathbb{R})$ corresponding to the inhomogeneous symplectic linear group.
$(S,[\ ,\ ]_\theta) \sim (S,\{\ ,\ \})$ is the maximal Lie algebra acting upon both $\mathcal{F}(\mathbb{R}^4)$ and $\mathcal{M}^\theta$ in terms of derivations.

- Any 3d Lie algebra $\mathfrak{g}$ with $[x_a, x_b] = f_{ab}{}^c x_c$ is isomorphic to

$$[x_1, x_2] = cx_3 + hx_2, \qquad [x_2, x_3] = ax_1, \qquad [x_3, x_1] = bx_2 - hx_3$$

with real parameters a, b, c, h such that $ah = 0$.

- The (classical) Jordan - Schwinger map $\pi_{\mathfrak{g}} : \mathbb{R}^4 \to \mathfrak{g}^* \sim \mathbb{R}^3$ can be defined such that

$$\{\pi^*_{\mathfrak{g}}(x_a), \pi^*_{\mathfrak{g}}(x_b)\} = f_{ab}{}^c \pi^*_{\mathfrak{g}}(x_c).$$

The Jordan - Schwinger map $\pi^*_{\mathfrak{g}}$ ranges within $\mathcal{P}_1 \oplus \mathcal{P}_2 \subset \mathcal{S}$.

- A (quantum, i.e. noncommutative) version of the J.S. map is the vector space inclusion $s_{\mathfrak{g}} : \mathfrak{g}^* \hookrightarrow \mathcal{P}_1 \oplus \mathcal{P}_2$ such that

$$[s_{\mathfrak{g}}(x_a), s_{\mathfrak{g}}(x_b)]_\theta = i\theta f_{ab}{}^c s_{\mathfrak{g}}(x_c).$$

- For a given 3d Lie algebra $\mathfrak{g}$, the Moyal product in $\mathbb{R}^4$ between $s_{\mathfrak{g}}(x_a)$ depend only on the $s_{\mathfrak{g}}(x_a)$ variables, so there exists a unital complex $*$-algebra $A_{\mathfrak{g}} \subset \mathcal{M}^\theta$ which is given as the quotient

$$A_{\mathfrak{g}} = [u_1, u_2, u_3]/I_{\mathfrak{g}} :$$

we are realizing the universal envelopping algebra $A_{\mathfrak{g}}$ as a subalgebra of $\mathcal{M}^\theta$.

- We list (some of the) maps $s_{\mathfrak{g}}$, starting by those corresponding to $a \neq 0$. They have a quadratic Casimir function $C_{\mathfrak{g}}$.

- For $\mathfrak{g} = \mathfrak{su}(2)$ it is $[x_a, x_b] = \varepsilon_{ab}{}^c x_c$, so $A_{\mathfrak{su}(2)} \subset \mathcal{M}^\theta$ is generated by

$$u_1 = \frac{1}{2}(q_1 q_2 + p_1 p_2), \qquad u_2 = \frac{1}{2}(q_1 p_2 - q_2 p_1), \qquad u_3 = \frac{1}{4}(q_1^2 + p_1^2 - q_2^2 - p_2^2).$$

The Casimir function is $C_{\mathfrak{su}(2)} = u_1^2 + u_2^2 + u_3^2$ with

$$u_4^2 = u_1^2 + u_2^2 + u_3^2, \qquad \text{with} \quad u_4 = \frac{1}{4}(q_1^2 + p_1^2 + q_2^2 + p_2^2),$$

- For $\mathfrak{g} = \mathfrak{e}(2)$ it is

$$[x_1, x_2] = x_3, \qquad [x_2, x_3] = 0, \qquad [x_3, x_1] = x_2$$

and we have $A_{\mathfrak{e}(2)} \subset \mathcal{M}^\theta$ generated by

$$u_1 = q_1 p_2 - q_2 p_1, \qquad u_2 = q_1, \qquad u_3 = q_2.$$

The quadratic Casimir function is $C_{\mathfrak{e}(2)} = (u_2^2 + u_3^2)/2$.

- For $\mathfrak{g} = \mathfrak{h}(1)$ (the Heseinberg-Weyl Lie algebra), with

$$[x_1, x_2] = x_3, \qquad [x_2, x_3] = 0, \qquad [x_3, x_1] = 0,$$

and we have $A_{\mathfrak{h}(1)} \subset \mathcal{M}^\theta$ generated by

$$u_1 = q_1, \qquad u_2 = q_2 p_1, \qquad u_3 = q_2.$$

The quadratic Casimir function is $C_{\mathfrak{h}(1)} = u_3^2$.

- A derivation D for an algebra A is inner if $Da = [f_D, a]$, otherwise it is outer. All derivations for the Moyal algebra $\mathcal{M}^\theta$ are inner.

- For any algebra $A_{\mathfrak{g}} \subset \mathcal{M}^\theta$, the union of its inner and outer derivations close a Lie algebra $\tilde{\mathfrak{g}}$ with $\mathfrak{g} \subseteq \tilde{\mathfrak{g}} \subset \mathfrak{isp}(4, \mathbb{R})$. The Lie algebra $\tilde{\mathfrak{g}}$ is seen to act via inner derivations upon $\mathcal{M}^\theta$, and such action can be projected onto $A_{\mathfrak{g}}$.

- The set $C_\wedge(\tilde{\mathfrak{g}}, A_{\mathfrak{g}})$ can be then described as a graded subalgebra of $C_\wedge(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta)$, the corresponding calculus $(C_\wedge(\tilde{\mathfrak{g}}, A_{\mathfrak{g}}), \mathrm{d})$ as a reduction of $(C_\wedge(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta), \mathrm{d})$.

- Moreover, the differential calculus that we define on $A_{\mathfrak{g}}$ turns out to have a frame, i.e. the exterior algebra is a free $A_{\mathfrak{g}}$-bimodule: this gives a way to study its cohomology.

- Since the structure of the space of derivations for $A_{\mathfrak{g}}$ strongly depends on the Lie algebra $\mathfrak{g}$ being semisimple or not, this talk will consider the two cases separately.

Differential calculus on $A_{\mathfrak{g}}$ for semisimple $\mathfrak{g} = \mathfrak{su}(2)$

- For a semisimple $\mathfrak{g}$, all derivations for $A_{\mathfrak{g}}$ are inner.

- The functions u_1, u_2, u_3 in $\mathbb{R}^4$ close $\{u_a, u_b\} = \epsilon_{abc} u_c$, with Casimir

$$u_4^2 = u_1^2 + u_2^2 + u_3^2, \qquad \text{with} \quad u_4 = \frac{1}{4}(q_1^2 + p_1^2 + q_2^2 + p_2^2),$$

  the corresponding Hamiltonian vector fields (X_1, X_2, X_3) give the right invariant vector fields tangent to S^3, while

$$X_4 = u_4^{-1} \sum_{j=1}^{3} u_j X_j$$

- We define the algebra $\tilde{A}$ as

$$\tilde{A} = \{f \in \mathcal{M}^\theta : [u_4, f]_\theta = 0\}.$$

  which slightly extends $A_{\mathfrak{su}(2)}$ since it contains the odd powers of u_4.

- The algebra $\tilde{A}$ is a noncommutative deformation of the commutative algebra $\mathcal{F}(\mathbb{R}^3 \backslash \{0\}) = f \in \mathcal{F}(\mathbb{R}^4 \backslash \{0\}) : L_{X_4} f = 0$.

- Within the classical setting, the rank of the space of derivations for A is 3, while the 4 derivations for $\tilde{A} \subset \mathcal{M}^\theta$ which are in $\mathfrak{isp}(4, \mathbb{R})$.

$$D_\mu(f) \ = \ [u_\mu, f]_\theta, \qquad \mu = 1, \dots, 4$$

are independent and give a 1d central extension $\tilde{\mathfrak{g}}$ of $\tilde{\mathfrak{g}} = \mathfrak{su}(2)$.

- The set $C^1_\wedge(\tilde{\mathfrak{g}}, \mathcal{M}^\theta) \subset C^1_\wedge(\mathfrak{isp}(4, \mathbb{R}), \mathcal{M}^\theta)$ contains the elements

$$\begin{aligned}
\alpha_1 &= \ p_2 * \mathrm{d}q_1 + p_1 * \mathrm{d}q_2 - q_2 * \mathrm{d}p_1 - q_1 * \mathrm{d}p_2, \\
\alpha_2 &= \ -q_2 * \mathrm{d}q_1 + q_1 * \mathrm{d}q_2 - p_2 * \mathrm{d}p_1 + p_1 * \mathrm{d}p_2, \\
\alpha_3 &= \ p_1 * \mathrm{d}q_1 - p_2 * \mathrm{d}q_2 - q_1 * \mathrm{d}p_1 + q_2 * \mathrm{d}p_2, \\
\beta &= \ q_1 * \mathrm{d}q_1 + q_2 * \mathrm{d}q_2 + p_1 * \mathrm{d}p_1 + p_2 * \mathrm{d}p_2
\end{aligned}$$

which satisfy the identities $(j, k = 1, \dots, 3)$

$$\begin{aligned}
\alpha_j(D_k) &= \ -2i\theta\, \delta_{jk} u_4, & \qquad \alpha_j(D_4) &= \ -2i\theta\, u_j \\
\beta(D_k) &= \ 0, & \beta(D_4) &= \ \theta^2
\end{aligned}$$

- Upon defining

$$\omega_j = \frac{i}{2\theta}\alpha_j - \frac{1}{\theta^2}u_k\beta, \qquad \omega_4 = \frac{1}{\theta^2}u_4\beta$$

we have $(\mu, \sigma = 1, \ldots, 4)$

$$\omega_\mu(D_\sigma) = u_4\delta_{\mu\sigma}.$$

- Since $u_4 \in Z(\tilde{A})$, we extend $\tilde{A}$ upon a localization, i.e. we define the element u_4^{-1} via the relations $u_4^{-1}u_4 = u_4u_4^{-1} = 1$ and $u_4^{-1}u_k = u_ku_4^{-1}$ for $k = 1, \ldots, 3$.

- The vector space $\mathcal{D} \simeq \tilde{\mathfrak{g}}$ is the tangent space to the noncommutative space described by the algebra $\tilde{A}$. The elements

$$\varphi_\mu = u_4^{-1}\omega_\mu$$

provide a basis for $\mathcal{D}^*$.
The action of the exterior derivative upon $\tilde{A}$ is given by

$$\mathrm{d}f = (D_\mu f)\varphi_\mu$$

where $D_\mu f = [u_\mu, f]_\theta$. The exterior algebra is defined as we saw.

- For this calculus we have

$$\begin{aligned} f * \varphi_\mu &= \varphi_\mu * f, \\ \varphi_\mu \wedge \varphi_\sigma &= -\varphi_\sigma \wedge \varphi_\mu \\ \mathrm{d}\varphi_j &= -\frac{1}{2}\varepsilon_{jkl}\varphi_k \wedge \varphi_l \qquad\qquad (j,k,l \in 1,\ldots,3) \\ \mathrm{d}\varphi_4 &= 0. \end{aligned}$$

  The Maurer-Cartan equation for the differential calculus depends on $\tilde{\mathfrak{g}}$, and its cohomology is related to the Eilenberg-Chevalley cohomology for $\tilde{\mathfrak{g}}$.

- Notice that the elements φ_a cannot be realised as $\sum_{a=1}^{3} f_a \mathrm{d}u_a$, so $C_\wedge(\tilde{\mathfrak{g}}, \tilde{A})$ extends the differential calculus $(\Omega_{\tilde{\mathfrak{g}}}, \mathrm{d})$ given as the smallest graded differential subalgebra of $C_\wedge(\tilde{\mathfrak{g}}, \tilde{A})$ generated in degree 0 by $\tilde{A}$ as described in the introduction.

- The Lie algebra $\mathfrak{g} = \mathfrak{e}(2)$ is

$$[x_1, x_2] = x_3, \qquad [x_2, x_3] = 0, \qquad [x_3, x_1] = x_2.$$

The Jordan - Schwinger map is given by

$$u_1 = q_1 p_2 - q_2 p_1, \qquad u_2 = q_1, \qquad u_3 = q_2.$$

The quadratic Casimir function is $u_C = (u_2^2 + u_3^2)/2 = (q_1^2 + q_2^2)/2$. The algebra is

$$A_{\mathfrak{e}(2)} = \{f \in \mathcal{M}^\theta : [u_C, f]_\theta = 0\}.$$

- Among the elements in $\mathfrak{isp}(4, \mathbb{R}) = (\mathcal{S}, [\ ,\ ]_\theta)$, the algebra $A_{\mathfrak{e}(2)}$ has inner derivations corresponding to elements u_a given above, and one exterior derivation

$$u_E = -(q_1 p_1 + q_2 p_2), \qquad D_E f = [u_E, f]_\theta, \qquad f \in A_{\mathfrak{e}(2)}$$

- This means that the action of the outer derivation D_E for $\tilde{A}$ can be represented as a commutator on $\tilde{A} \subset \mathcal{M}^\theta$ as an inner derivation that can be projected.
  The element u_E is defined up to an arbitrary function of the quadratic Casimir u_C, but this does not affect any of the results we shall describe.

- The Lie algebra of derivations for $A_{\mathfrak{e}(2)}$ is then given by $\tilde{e}(2)$ spanned by $\{u_1, u_2, u_3, u_E\}$ w.r.t. the $*$-product in $\mathcal{M}^\theta$. It is a 1d extension of $\mathfrak{e}(2)$ as

$$[u_\mu, u_\nu]_\theta \;=\; i\theta \tilde{f}_{\mu\nu}{}^{\rho} u_\rho, \qquad\qquad \mu, \nu, \rho = 1, \ldots, 4.$$

- The set $\mathcal{D} \simeq \tilde{\mathfrak{e}}(2)$ gives the tangent space to the differential calculus. Since $\tilde{\mathfrak{e}}(2) \subset \mathfrak{isp}(4, \mathbb{R})$, we consider the elements $\alpha_\mu = \mathrm{d}u_\mu$.

- The elements in $C^1_\wedge(\mathfrak{isp}(4,\mathbb{R}),\mathcal{M}^\theta)$ given by

$$\begin{aligned}\omega_1 &= \frac{1}{2}(u_3\alpha_2 - u_2\alpha_3),\\ \omega_2 &= -\frac{1}{2}(u_3\alpha_1 + u_2\alpha_E),\\ \omega_3 &= \frac{1}{2}(u_2\alpha_1 - u_3\alpha_E),\\ \omega_E &= \frac{1}{2}(u_2\alpha_2 + u_3\alpha_3)\end{aligned}$$

verify (with $\mu = 1,\ldots,4$)

$$\omega_\mu(D_\sigma) = (i\theta)u_C\delta_{\mu\sigma},$$

so the elements

$$\varphi_\mu = -\frac{i}{\theta}\,u_C^{-1}\omega_\mu$$

give a basis for $C^1_\wedge(\tilde{\mathfrak{e}}(2),A_{\mathfrak{e}(2)})$ (after the natural localisation).

• Analogously to the previous case one has

$$
\begin{aligned}
f * \varphi_\mu &= \varphi_\mu * f, & \varphi_\mu \wedge \varphi_\sigma &= -\varphi_\sigma \wedge \varphi_\mu, \\
\mathrm{d} f &= (D_\mu f)\varphi_\mu = ([u_\mu, f]_\theta)\varphi_\mu, & \mathrm{d}\varphi_\rho &= -\frac{1}{2} i\theta \tilde{f}_{\mu\nu}{}^{\rho} \varphi_\mu \wedge \varphi_\nu
\end{aligned}
$$

• The presence of a 1-form which dualises the outer derivation for $A_{\mathfrak{e}(2)}$ means that centre $Z(A_{\mathfrak{e}(2)})$ is not in the kernel of the d operator.

$$
\mathrm{d} u_C = 2(i\theta) u_C \varphi_4.
$$

• Also in this case we see that the 1 forms φ_μ can not be written as $f_a \mathrm{d} g_a$ using only elements in $A_{\mathfrak{e}(2)}$.

• Since we localised the algebra upon adding the generator u_C^{-1}, we have defined a differential calculus on the algebra $A_{\mathfrak{e}(2)}$ deforming the classical algebra $\mathcal{F}(\mathbb{R}^3 \backslash (x_1^2 + x_2^2 = 0))$.

- For $\mathfrak{g} = \mathfrak{h}(1)$ (the Heseinberg-Weyl Lie algebra), with

$$[x_1, x_2] = x_3, \qquad [x_2, x_3] = 0, \qquad [x_3, x_1] = 0,$$

and we have $A_{\mathfrak{h}(1)} \subset \mathcal{M}^\theta$ generated by

$$u_1 = q_1, \qquad u_2 = q_2 p_1, \qquad u_3 = q_2.$$

The quadratic Casimir function is $C_{\mathfrak{h}(1)} = u_3^2$. The algebra is

$$A_{\mathfrak{h}(1)} = \{f \in \mathcal{M}^\theta : [u_3, f]_\theta = 0\}.$$

- Among the elements in $\mathfrak{isp}(4, \mathbb{R}) = (\mathcal{S}, [\ ,\ ])$, the algebra $A_{\mathfrak{h}(1)}$ has inner derivations corresponding to the elements u_a above, and exterior derivations given by $D_{E_a}(f) = [u_{E_a}, f]_\theta$ for $f \in A_{\mathfrak{h}(1)}$ with

$$u_{E_1} = -(p_1 q_1 + p_2 q_2), \qquad u_{E_2} = -p_2 q_2.$$

- The action of the outer derivation D_{E_a} for $A_{\mathfrak{h}(1)}$ can be represented as a commutator on $A_{\mathfrak{h}(1)} \subset \mathcal{M}^\theta$ in terms of the quadratic element $u_{E_a} \in S \subset \mathcal{M}^\theta$.

- We then span a tangent space for of derivations for a differential calculus on $A_{\mathfrak{h}(1)}$

$$D_\sigma(f) = [u_\sigma, f]_\theta$$

with $\{u_\sigma\}_{\sigma=1,\ldots,4} = \{u_1, u_2, u_3, u_4 = -\mu p_1 q_1 - \nu p_2 q_2\}$, with $\mu, \nu \in \mathbb{R}$.

They close the Lie algebra $\tilde{\mathfrak{h}}(1) = \{\mathfrak{h}(1),\ \mu E_1 + (\nu - \mu) E_2\}$ with

$$\begin{aligned}
[u_1, u_2]_\theta &= (i\theta) u_3, \\
[u_1, u_4]_\theta &= -(i\theta)\mu u_1, \\
[u_2, u_4]_\theta &= (i\theta)(\mu - \nu) u_2, \\
[u_3, u_4]_\theta &= -(i\theta)\nu u_3.
\end{aligned}$$

• Since $\tilde{\mathfrak{h}}(1) \subset \mathfrak{isp}(4,\mathbb{R})$, we consider the elements $\{\alpha_\rho = \mathrm{d}u_\rho\}_{\rho=1,\dots,4} \in (C_\wedge(\mathfrak{isp}(4,\mathbb{R}), \mathcal{M}^\theta), \mathrm{d})$,
and see that, if $\nu \neq 0$, the elements

$$\begin{aligned}
\omega_1 &= \frac{1}{2}(u_3\alpha_2 + (\frac{\mu}{\nu} - 1)u_2\alpha_3), \\
\omega_2 &= \frac{1}{2}(-u_3\alpha_1 + \frac{\mu}{\nu}u_1\alpha_3), \\
\omega_3 &= \frac{1}{2}((1 - \frac{\mu}{\nu})u_2\alpha_1 - \frac{\mu}{\nu}u_1\alpha_2 + (i\theta)(\frac{\mu}{\nu} - \frac{\mu^2}{\nu^2})\alpha_3 - \frac{1}{\nu}u_3\alpha_4), \\
\omega_4 &= \frac{1}{2\nu}u_3\alpha_3
\end{aligned}$$

verify

$$\omega_\rho(D_\sigma) = (i\theta)u_3\delta_{\rho\sigma}.$$

- After the usual localisation, by adding u_3^{-1} corresponding to the Casimir function, the elements

$$\varphi_\rho = -\frac{i}{\theta} u_3^{-1} \omega_\rho$$

give a basis for $C^1_\wedge(\tilde{\mathfrak{h}}(1), A_{\mathfrak{h}(1)})$. One has

$$\mathrm{d}f = (D_\rho f)\varphi_\rho = ([u_\rho, f]_\theta)\varphi_\rho.$$

and

$$\mathrm{d}u_3 = (i\theta)\nu u_3 \, \varphi_4,$$

thus proving that the centre $Z(A_{\mathfrak{h}(1)})$ of the algebra is not in the kernel of the exterior derivative d.

- Since we localised the algebra upon adding the generator u_C^{-1}, we have defined a differential calculus on the algebra $A_{\mathfrak{h}(1)}$ deforming the classical algebra $\mathcal{F}(\mathbb{R}^3 \backslash (x_3 = 0))$.

Conclusion

- Our analysis brought to a 4 differential calculus on the algebras $A_{\mathfrak{g}} \subset \mathcal{M}^{\theta}$. Such algebras deform spaces which are classically 3d. Such classical spaces are the foliations of the codimension one regular orbits for the action of the Lie algebra $\mathfrak{g}$ upon $\mathfrak{g}^* \simeq \mathbb{R}^3$.

- Our analysis works for 3d Lie algebras having a global Casimir quadratic functions. It does not apply to the case $\mathfrak{g} = \mathfrak{sb}(2, \mathbb{C})$ which gives the so called κ-Minkowski space: we are working on it.

- Since we have a (global) frame for the calculi on $A_{\mathfrak{g}}$, we can define symmetric forms on it (say metrics), spinors, Hodge and Laplacians. The problem we are concerned with at the moment is that only for semisimple $\mathfrak{g}$ there exists a natural invariant metric.

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