Contact Lagrangian structures and dissipation

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What is dissipation?

Newtonian Picture
The interaction of the system with an environment is expressed by means of forces which in general are not "conservative". What are called "dissipative forces" are usually an effective way to take into account the coupling with the environment without the consideration of additional degrees of freedom with respect to those possessed by the system we are considering.

For example, the introduction of explicit "time dependent forces" is a manner to take into account the effect of the environment and posible interactions.

We want to stress that if we want to declare a system to be dissipative, we should say what is being dissipated.

When we are simply given a dynamical system whose dynamics is described by means of a vector field, say a second order vector field. A major question is whether it is possible to characterize the dynamics as a "conservative" or "dissipative"

Lagrangian Picture

In order to give a meaningful characterization one may procede looking for a possible Lagrangian description of the given second order vector field. Then, if a Lagrangian description exists, in the time-independent case we would say that the system preserves the "Lagrangian energy" function

$$\mathcal{E}_{\mathfrak{L}} = \dot{q}_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j} - \mathfrak{L}\,, \tag{1}$$

associated with the Lagrangian $\mathfrak{L}$ for the second order vector field Γ, which need not coincide the "physical energy" E of the system.

Therefore, we could qualify the system to be dissipative even if it admits a description by means of a Lagrangian.

Example.

We consider a second order differential equation on $TQ = \mathbb{R} \times \mathbb{R}$ with a friction force proportional to the velocity by means of a friction coefficient γ, i.e.:

$$\ddot{q} + \gamma\, \dot{q} = 0\,. \tag{2}$$

A possible (local) Lagrangian for this system with its corresponding Lagrangian energy are:

$$\mathfrak{L} = \dot{q}(\ln \dot{q}) - \gamma\, q \quad \text{and} \quad \mathcal{E}_{\mathfrak{L}} = \dot{q} + \gamma\, q \tag{3}$$

respectively. The rate of dissipation of the mechanical energy $E = \frac{m}{2}\dot{q}^2$ is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\gamma\, \dot{q}^2. \tag{4}$$

When the system does not admit of a Lagrangian description, we have to develop new strategies in order to characterize dissipation. In order to explain one such a strategy, let us consider the physically relevant situation represented by linear dynamical systems

$$m^{jk}\ddot{q}_k + \gamma^{jk}\dot{q}_k + \omega^{jk}q_k = 0\,. \tag{5}$$

When $||m^{jk}||$ is non-degenerate the differential equation defines a second order vector field. The linear vector field Γ associated with the equations of motion on the linear manifold $T\mathbb{R}^n$, and the representative matrix G^j_k of Γ defined by:

$$\Gamma \;=\; G^j_k\,\xi^k\,\frac{\partial}{\partial \xi^j}\,, \tag{6}$$

where $\{\xi^j\}_{j=1,\ldots,2n}$ is a collective Cartesian coordinates system on $T\mathbb{R}^n$.

The system may be given a description by means of a constant Poisson structure Λ on $T\mathbb{R}^n$, represented in the coordinate system $\{\xi^j\}_{j=1,\ldots,2n}$ by an antisymmetric numerical matrix $||\Lambda^{jk}||$, and a quadratic Hamiltonian function:

$$H = \frac{1}{2} H_{jk} \xi^j \xi^k \,, \tag{7}$$

if and only if the representative matrix $||G^j_k||$ is traceless with all its odd powers[1], namely:

$$\mathrm{Tr}\{G^{2k+1}\} = 0 \quad \text{for all } k \,. \tag{8}$$

[1] Giordano, M., Marmo, G. and Rubano, C., 1993. The inverse problem in the Hamiltonian formalism: integrability of linear Hamiltonian fields. Inverse Problems, 9(4), p.443.

Two coupled oscillations with different frequencies ω_k, different damping coefficients γ_k, i.e.

$$\begin{aligned} \ddot{q}_1 + \gamma_1 \dot{q}_1 + \omega_1^2 q_1 + \kappa q_2 + \delta \dot{q}_2 = 0 \\ \ddot{q}_2 + \gamma_2 \dot{q}_2 + \omega_2^2 q_2 + \kappa q_1 + \delta \dot{q}_1 = 0 \,. \end{aligned} \tag{9}$$

Then, the representative matrix is:

$$G = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\omega_1^2 & -\kappa & -\gamma_1 & -\delta \\ -\kappa & -\omega_2^2 & -\delta & -\gamma_2 \end{pmatrix} . \tag{10}$$

We may, in full generality, prove that this system does not allow for any Lagrangian description

It is easy to see that the representative matrix can be decomposed as

$$||G^j_k|| = ||A^j_k|| + ||D^j_k|| \,, \tag{11}$$

where $||A^j_k||$ is a traceless matrix, clearly the decomposition is arbitrary. Once a choice has been made, it is possible to think of the dynamical vector field as the sum of a "comparison" or "reference dynamics" with a "perturbation term".
The linear vector field Γ may be decomposed as:

$$\Gamma \;=\; A^j_k \, \xi^k \, \frac{\partial}{\partial \xi^j} + D^j_k \, \xi^k \, \frac{\partial}{\partial \xi^j} \,, \tag{12}$$

the "comparison dynamics" admitting a Hamiltonian description, while the second term may be thought of as the perturbation term responsible for the dissipation of the Hamiltonian energy function associated with the comparison dynamics.

Other very interesting examples arise from the geometrical description of the evolution of density states for finite level quantum systems, i.e., vector fields associated with Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) master equations[2].
Here, if we start with a GKLS equation, we would have a decomposition into Hamiltonian vector field $X_{\mathbf{H}}$, a Gradient vector field $Y_{\mathbf{V}}$ and a Jump vector field $Z_{\mathcal{K}}$ as follows:

$$\Gamma = X_{\mathbf{H}} - Y_{\mathbf{V}} + Z_{\mathcal{K}} \,. \tag{13}$$

$X_{\mathbf{H}}$ preserves the spectrum of the quantum states;
$Y_{\mathbf{V}}$ does not preserve the spectrum but preserves the rank;
$Z_{\mathcal{K}}$ does not preserve the spectrum nor the rank of quantum states.

[2]Ciaglia, F.M., Di Cosmo, F., Ibort, A., Laudato, M. and Marmo, G., 2017. Dynamical vector fields on the manifold of Quantum states. Open Systems and Information Dynamics, 24(03), p.1740003.

Contact Lagrangian structures

Once a "decomposition principle" for the system has been defined, it is possible to go to a contact formalism to show that a "dissipative" system may be described in terms of a "contact Hamiltonian formalism"[3].

An exact contact manifold is defined as a differential odd-dimensional manifold admitting a global differential contact 1-form η such that the following condition

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0\,, \tag{14}$$

holds everywhere.

Associated with the definition of the contact 1-form on a contact manifold, there is another fundamental object, called the *Reeb vector field* ξ, which is defined by the conditions

$$i_\xi \eta = 1, \qquad i_\xi \mathrm{d}\eta = 0\,. \tag{15}$$

[3] Bravetti, A., Cruz, H. and Tapias, D., 2017. Contact hamiltonian mechanics. Annals of Physics, 376, pp.17-39.

Now, assuming that the carrier space for the dynamics is $TQ \times \mathbb{R}$ with an exact contact structure (η, ξ), in order to define the dynamics on $TQ \times \mathbb{R}$ one can associate with every smooth function $\mathscr{E}$ a vector field Γ_c on $TQ \times \mathbb{R}$ by means of:

$$i_{\Gamma_c} \mathrm{d}\eta = \mathrm{d}\mathscr{E} - (\pounds_\xi \mathscr{E})\eta \qquad \text{and} \qquad i_{\Gamma_c}\eta = -\mathscr{E}\,, \tag{16}$$

where $\mathscr{E}$ is called the "contact Lagrangian energy".
In particular, we assume that, locally, the 1-form η can be written as:

$$\eta = \mathrm{d}S - \theta_{\mathfrak{L}} \quad \text{with} \quad \theta_{\mathfrak{L}} = \mathrm{d}q_j \frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\,, \tag{17}$$

where $(q_j,\ \dot{q}_j,\ S)$ are local coordinates on $TQ \times \mathbb{R}$, $\mathfrak{L}$ is the Lagrangian function of the "comparison system", and the contact Lagrangian energy $\mathscr{E} \equiv \mathscr{E}_{\mathfrak{L}}$ may be written as:

$$\mathscr{E}_{\mathfrak{L}} = \mathcal{E}_{\mathfrak{L}} + h(S)\,. \tag{18}$$

Introducing these definitions into the conditions in (16) we obtain the Euler–Lagrange equations and the equation for the component of the vector field in the direction of the Reeb vector, namely

$$\pounds_{\Gamma_C}\theta_{\mathfrak{L}} - \mathrm{d}\mathfrak{L} = -\frac{\mathrm{d}h}{\mathrm{d}S}\,\theta_{\mathfrak{L}}\,, \tag{19}$$

$$\dot{S} = i_{\Gamma_C}\theta_{\mathfrak{L}} - \mathscr{E}_{\mathfrak{L}}\,. \tag{20}$$

Equivalently, using the definition for the 1-form $\theta_{\mathfrak{L}}$ and spelling the Lie derivative on $\theta_{\mathfrak{L}}$ it is easy to obtain the coordinate expression of the Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathfrak{L}}{\partial\dot{q}_j} - \frac{\partial\mathfrak{L}}{\partial q_j} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial\mathfrak{L}}{\partial\dot{q}_j}\,, \tag{21}$$

which in general, these are implicit differential equations. In addition, we have in coordinates that

$$\dot{S} = \mathfrak{L} - h(S)\,. \tag{22}$$

We may look at these systems as a sort of generalization of the so-called Caldirola-Kanai dissipative systems, where the Lagrangian energy is not preserved along the dynamical trajectories, indeed:

$$\frac{\mathrm{d}\mathcal{E}_{\mathfrak{L}}}{\mathrm{d}t} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\,, \tag{23}$$

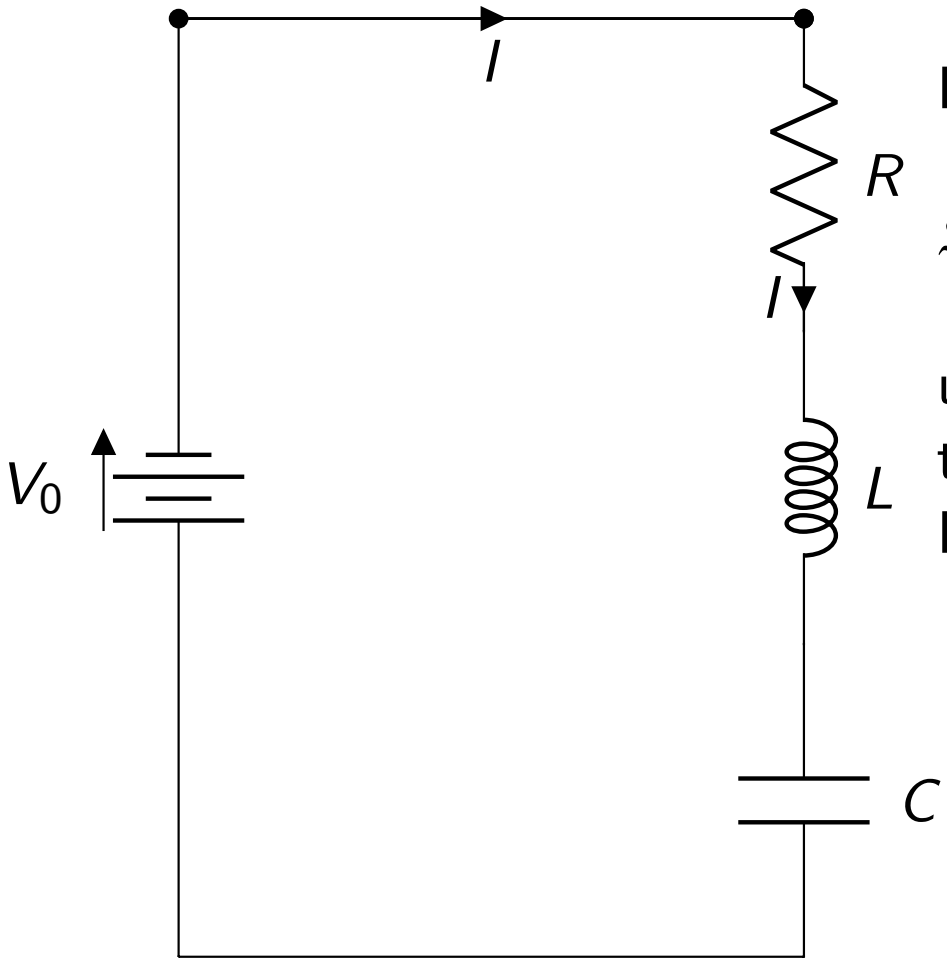
which may be positive or negative according to the sign of $\frac{\mathrm{d}h}{\mathrm{d}S}$.

Defined the contact dynamics $\Gamma_c \in \mathfrak{X}(TQ \times \mathbb{R})$ a natural question is whether it is possible to project such dynamics onto a second order vector field $\Gamma \in \mathfrak{X}(TQ)$. It is clear from contact Euler–Lagrange

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathfrak{L}}{\partial \dot{q}_j} - \frac{\partial \mathfrak{L}}{\partial q_j} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial \mathfrak{L}}{\partial \dot{q}_j}\,, \tag{24}$$

that it reproduces a second order dynamics if $h(S)$ is linear in S. [4]

[4] Grmela, M. and Öttinger, H.C., 1997. Dynamics and thermodynamics of complex fluids. I. Development of a general formalism. Physical Review E, 56(6), p.6620.



For RLC circuits we can consider

$$\mathfrak{L} = \frac{1}{2} L \dot{I}^2 - \frac{1}{2\,C} I^2 \quad \text{and} \quad h(S) = R\,S$$

using the notation indicated in the figure, the contact Euler–Lagrange equation

$$L\ddot{I} + R\dot{I} + \frac{1}{C} I = 0\,.$$

In the more general sense, a contact manifold is defined as an odd-dimensional differential manifold admitting a global 1-form η and a global 2-form ω such that

$$\eta \wedge \omega^n \neq 0 \tag{25}$$

everywhere. The $(2n+1)$-form $\Omega = \eta \wedge \omega^n$ provides a volume form.

Let us take our contact manifold to be again $TQ \times \mathbb{R}$. Then, we define a dynamical evolution in terms of a vector field associated with a smooth function $\mathscr{F}$ by means of

$$i_{\Gamma_C}\,\Omega = n\,(\mathrm{d}\mathscr{F} \wedge \eta) \wedge \omega^{n-1} + \mathscr{F}\,\omega^n\,. \tag{26}$$

Similarly to what we have done in the case of exact contact structures on $TQ \times \mathbb{R}$, here we assume that η and ω are given by

$$\eta = \mathrm{d}S - \alpha \qquad \text{and} \qquad \omega = -\,\mathrm{d}\theta_{\mathfrak{L}}\,, \tag{27}$$

where α is a semi-basic 1-form with the following local expression:

$$\alpha = \sum_{i=1}^{n} a_j(q_j, \dot{q}_j, S)\,\mathrm{d}q_j\,. \tag{28}$$

Becuase we are interested in characterizing dissipative systems in terms of a "decomposition principle", we shall assume that $\mathscr{F} = -\,\mathscr{E}_{\mathfrak{L}}$, where $\mathscr{E}_{\mathfrak{L}} = \mathcal{E}_{\mathfrak{L}} + h(S)$.

Using these definitions we get the conditions

$$\pounds_{\Gamma_C}\theta_{\mathfrak{L}} - \mathrm{d}\mathfrak{L} = -\frac{\mathrm{d}h}{\mathrm{d}S}\,\alpha \qquad \text{and} \qquad \dot{S} = i_{\Gamma_C}\alpha - \mathscr{E}_{\mathfrak{L}}\,. \tag{29}$$

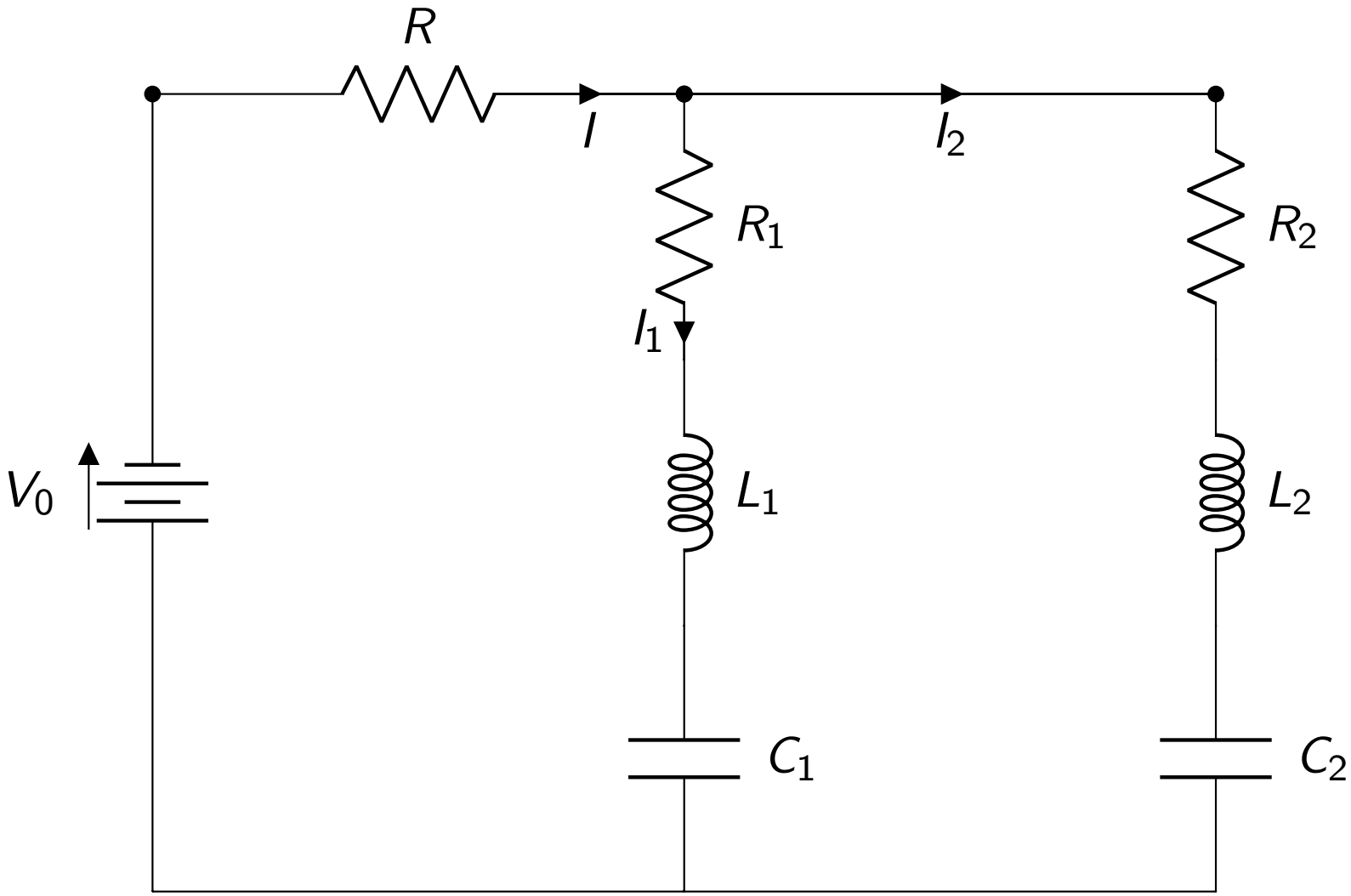
We may identify some special classes of dissipative systems by making qualifications for α. For instance, if we take:

$$\alpha = \mathrm{d}q_j \frac{\partial \mathcal{F}}{\partial \dot{q}_j}\,, \tag{30}$$

where $\mathcal{F}$ is an arbitrary velocity-dependent function. The contact Euler-Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathfrak{L}}{\partial \dot{q}_j} - \frac{\partial \mathfrak{L}}{\partial q_j} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial \mathcal{F}}{\partial \dot{q}_j}\,, \quad \frac{\mathrm{d}\mathcal{E}_{\mathfrak{L}}}{\mathrm{d}t} = -\frac{\mathrm{d}h}{\mathrm{d}S}\frac{\partial \mathcal{F}}{\partial \dot{q}_j}\,. \tag{31}$$

Example.



Then, the dynamics of the system is determined by:

$$\mathfrak{L} = \frac{1}{2} L^{jk} \dot{I}_k \dot{I}_j - \frac{1}{2} C^{jk} I_k I_j \,, \quad h(S) = S \quad \text{and} \quad \mathcal{F} = -\frac{1}{2} R^{jk} \dot{I}_k \dot{I}_j \,, \tag{32}$$

where here I_1 and I_2 denotes the currents in the branches, and according with the notation in the Figure the matrices are

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix} , \quad C = \begin{pmatrix} 1/C_1 & 0 \\ 0 & 1/C_2 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} R_1 & R \\ R & R_2 \end{pmatrix} . \tag{33}$$

Therefore, the contact Euler-Lagrange equations associated with this system corresponds to

$$L^{jk} \ddot{I}_k + R^{jk} \dot{I}_k + C^{jk} I_k = 0 \,, \tag{34}$$

which are in agreement with the equations obtained from the Kirchhoff's circuit laws. In addition, it is clear that if the coupling parameter $R \to 0$ the system is reduced to two non-interacting RLC circuits.

Conclusions and perspectives

- In conclusion, when we are simply given a dynamical system whose dynamics is described by means of a vector field, it does not make sense to say that the system is "conservative" or "dissipative" *per se* because dissipation is a relational concept.
- The contact formalism for the description of dissipative systems gives rise to a "decomposition principle", where the "conservative" part accepts a Lagrangian description and the "perturbation" is a contact corrections.
- A major question is whether it is possible define dissipation at the quantum level by means of the contact formalism.

Thanks!