

# On divisibility and quantum Markovianity for non-invertible dynamical maps

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# Dynamical map

Quantum system  $\longleftrightarrow \mathcal{H}$

quantum states  $\longleftrightarrow \rho \geq 0 , \text{Tr}\rho = 1$

Quantum evolution  $\longleftrightarrow \rho \rightarrow \rho_t := \Lambda_t(\rho)$

$\Lambda_t : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) ; t \geq 0$

- completely positive
- trace-preserving
- $\Lambda_{t=0} = \mathbb{1}$

# Positivity vs. Complete Positivity

$$\dim \mathcal{H} = d < \infty$$

$$\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$$

$$\text{Positivity } X \geq 0 \longrightarrow \Phi(X) \geq 0$$

$$\text{Complete Positivity} \longleftrightarrow \text{Positivity of } \text{id} \otimes \Phi$$

$$\text{Quantum channel} \longleftrightarrow \text{CPTP map}$$

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## Markovian semigroup

$$\frac{d}{dt} \Lambda_t = L \Lambda_t ; \quad \longrightarrow \quad \Lambda_t = e^{tL} ; \quad t \geq 0$$

What is the most general  $L$  ?

Theorem (Gorini-Kossakowski-Sudarshan-Lindblad (1976))

$\Lambda_t = e^{tL}$  is CPTP if and only if

$$L[\rho] = -i[H, \rho] + \frac{1}{2} \sum_{\alpha} \gamma_{\alpha} \left( [V_{\alpha}, \rho V_{\alpha}^{\dagger}] + [V_{\alpha} \rho, V_{\alpha}^{\dagger}] \right) ; \quad \gamma_{\alpha} > 0$$

## Beyond Markovian semigroup

- non-local master equation (Nakajima-Zwanzig equation)

$$\frac{d}{dt} \Lambda_t = \int_0^t K_{t-\tau} \Lambda_\tau d\tau$$

- local in time master equation

$$\frac{d}{dt} \Lambda_t = L_t \Lambda_t$$

# Divisibility vs Markovianity

$$\Lambda_t = V_{t,s} \circ \Lambda_s \quad ; \quad t \geq s$$

$$V_{t,s} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$$

- P-divisible iff  $V_{t,s}$  is PTP
- CP-divisible iff  $V_{t,s}$  is CPTP

CP-divisible  $\implies$  P-divisible

Theorem (Benatti, DC, Fillipov (2017))

$\Lambda_t$  is CP-divisible iff  $\Lambda_t \otimes \Lambda_t$  is P-divisible

Evolution is MARKOVIAN iff  $\Lambda_t$  is CP-divisible

## Markovian vs. non-Markovian

- Markovianity is defined for classical stochastic processes
- Markovianity = semigroup dynamics:  $\Lambda_t = e^{tL}$
- **Markovianity = CP-divisibility (Rivas, Huelga, Plenio)**
- Markovianity = negative information flow (Breuer, Laine, Piilo)
- Geometrical characterization of non-Markovianity (Lorenzo, Plastina, Paternostro)
- non-Markovianity via mutual information (Luo)
- non-Markovianity via channel capacity (Bylicka, DC, Maniscalco)
- non-Markovianity via channel discrimination (Bae, DC)
- ...

## Divisibility vs. monotonicity of trace norm

$$\|X\|_1 = \mathrm{Tr}\sqrt{XX^\dagger}$$

### Theorem

If  $\Lambda_t$  is  $P$ -divisible, then

$$\frac{d}{dt} \|\Lambda_t(X)\|_1 \leq 0$$

for any  $X = X^\dagger \in \mathcal{B}(\mathcal{H})$ .

## Divisibility vs. relative entropy

$$S(\rho||\sigma) = \text{Tr}(\rho[\log \rho - \log \sigma])$$

### Theorem

If  $\Lambda_t$  is  $P$ -divisible, then

$$\frac{d}{dt} S(\Lambda_t(\rho)||\Lambda_t(\sigma)) \leq 0$$

# Divisibility vs. Rényi sandwiched entropy

$$\tilde{D}_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \log \left( \text{Tr} \left[ \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right] \right)$$

$$\alpha \in (0, 1) \cup (1, \infty)$$

$$\tilde{D}_{\frac{1}{2}}(\rho||\sigma) = -2 \log F(\rho, \sigma)$$

## Theorem

If  $\Lambda_t$  is CP-divisible (P-divisible), then

$$\frac{d}{dt} \tilde{D}_\alpha(\Lambda_t(\rho)||\Lambda_t(\sigma)) \leq 0$$

$$\alpha \in [\frac{1}{2}, 1) \cup (1, \infty) \quad \left( \left\{ \frac{1}{2} \right\} \cup (1, \infty) \right)$$

# Divisibility vs. distinguishability of quantum states

$$P_{\text{guess}} = \frac{1}{2} \left( 1 + \frac{1}{2} \|\rho_1 - \rho_2\|_1 \right)$$

## Theorem

If  $\Lambda_t$  is P-divisible, then

$$\frac{d}{dt} \|\Lambda_t(\rho_1 - \rho_2)\|_1 \leq 0$$

for all pair  $\rho_1, \rho_2$ .

Breuer, Laine, Piilo (BLP)  $\longrightarrow$  Markovianity

P-divisibility  $\implies$  BLP condition

## Invertible maps

$$\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$$

$$\Phi^{-1} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$$

If  $\Phi$  is trace-preserving the inverse  $\Phi^{-1}$  is trace-preserving

If  $\Phi$  is positive (CP) the inverse  $\Phi^{-1}$  needs NOT be positive (CP) !!!

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## Invertible maps

$$\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \quad \text{--- PTP}$$

### Theorem

$\Phi^{-1}$  is PTP iff

$$\Phi(X) = UXU^\dagger \quad \text{or} \quad \Phi(X) = UX^T U^\dagger$$

$$\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \quad \text{--- CPTP}$$

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$$\{p_1, \rho_1; p_2, \rho_2\} \longrightarrow \frac{1}{2} (1 + \|p_1\rho_1 - p_2\rho_2\|_1)$$

### Theorem (DC, Kossakowski, Rivas)

If  $\Lambda_t$  is invertible, then it is P-divisible iff

$$\frac{d}{dt} \|\Lambda_t(p_1\rho_1 - p_2\rho_2)\|_1 \leq 0$$

for all pair  $\rho_1, \rho_2$  and  $p_1, p_2$ .

It is CP-divisible iff

$$\frac{d}{dt} \|\mathbb{1} \otimes \Lambda_t(p_1\tilde{\rho}_1 - p_2\tilde{\rho}_2)\|_1 \leq 0$$

for all pair  $\tilde{\rho}_1, \tilde{\rho}_2$  and  $p_1, p_2$ .

$$\{p_1, \rho_1; p_2, \rho_2; \dots; p_n, \rho_n\} \longrightarrow P_{\text{guess}}^{(n)} = \max_{E_i} \sum_{i=1}^n p_i \text{Tr}(E_i \rho_i)$$

$$P_{\text{guess}}^{(n)}(t) = \max_{E_i} \sum_{i=1}^n p_i \text{Tr}[E_i \Lambda_t(\rho_i)]$$

### Theorem

If  $\Lambda_t$  is  $P$ -divisible, then

$$\frac{d}{dt} P_{\text{guess}}^{(n)}(t) \leq 0$$

for all ensembles.

Hence, if  $\Lambda_t$  is invertible, it is enough to check  $n = 2$  only!

## Master equation

$$\frac{d}{dt} \Lambda_t = L_t \Lambda_t , \quad \Lambda_0 = \mathbb{1}$$

### Theorem

If  $\Lambda_t$  is invertible, then it is CP-divisible iff

$$L_t(\rho) = -i[H(t), \rho] + \sum_{\alpha} \gamma_{\alpha}(t) \left( [V_{\alpha}(t), \rho V_{\alpha}^{\dagger}(t)] + [V_{\alpha}(t)\rho, V_{\alpha}^{\dagger}(t)] \right)$$

and  $\gamma_{\alpha}(t) \geq 0$ .

Example:

$$L_t \rho = \frac{1}{2} \sum_{k=1}^3 \gamma_k(t) [\sigma_k \rho \sigma_k - \rho] = \sum_k \gamma_k(t) L_k$$

$$\Lambda_t \text{ is invertible} \iff \Gamma_k(t) = \int_0^t \gamma_k(\tau) d\tau < \infty$$

- $\Lambda_t$  is CP-divisible iff

$$\gamma_1(t) \geq 0 ; \quad \gamma_2(t) \geq 0 ; \quad \gamma_3(t) \geq 0$$

- $\Lambda_t$  is P-divisible iff

$$\gamma_1(t) + \gamma_2(t) \geq 0 ; \quad \gamma_1(t) + \gamma_3(t) \geq 0 ; \quad \gamma_2(t) + \gamma_3(t) \geq 0$$

## “Eternal non-Markovianity”

(Hall, Cresser, Li, Andersson, PRA 2014)

$$\gamma_1(t) = \gamma_2(t) = 1 ; \quad \gamma_3(t) = -\tan t < 0$$

$$\dot{\Lambda}_t = L_t \Lambda_t \longrightarrow \Lambda_t$$

- $\Lambda_t$  is CPTP
- $\Lambda_t$  is not CP-divisible
- $\Lambda_t$  is P-divisible!

$$\gamma_1(t) + \gamma_2(t) \geq 0 ; \quad \gamma_1(t) + \gamma_3(t) \geq 0 ; \quad \gamma_2(t) + \gamma_3(t) \geq 0$$

$$\Lambda_t = \frac{1}{2}(e^{tL_1} + e^{tL_2})$$

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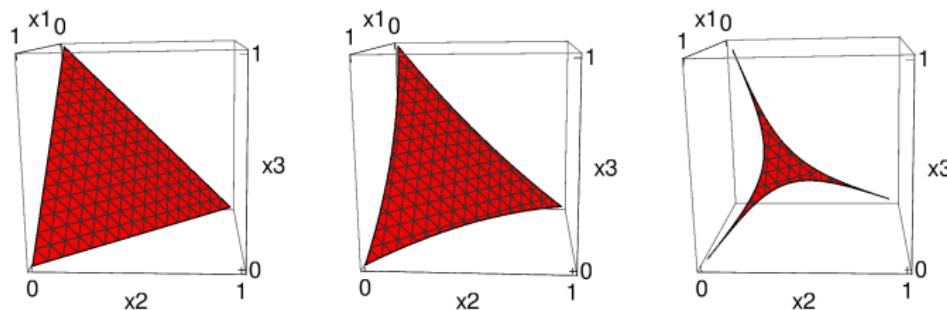
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- $\Lambda_t$  is P-divisible!

$$\gamma_1(t) + \gamma_2(t) \geq 0 ; \quad \gamma_1(t) + \gamma_3(t) \geq 0 ; \quad \gamma_2(t) + \gamma_3(t) \geq 0$$

$$\Lambda_t = \frac{1}{2}(e^{tL_1} + e^{tL_2})$$

$$\Lambda_t = x_1 e^{tL_1} + x_2 e^{tL_2} + x_3 e^{tL_3}$$

- $\Lambda_t$  is a Markovian semi-group if only one  $x_k = 1$
- $\Lambda_t$  is P-divisible for all  $x_k$
- $\Lambda_t$  is CP-divisible for “small region”



## Mutually Unbiased Bases (MUBs)

$$\dim \mathcal{H} = d$$

Two orthonormal bases  $|\psi_k\rangle$  and  $|\phi_l\rangle$  in  $\mathcal{H}$  are mutually unbiased

$$|\langle\psi_k|\phi_l\rangle|^2 = \frac{1}{d}$$

for all  $k$  and  $l$ .

# Mutually Unbiased Bases

$N(d)$  = maximal number of MUBs

- $N(d) \leq d + 1$
- if  $d = p^r$  ( $p$  prime), then  $N(d) = d + 1$
- if  $d \neq p^r$  ( $p$  prime) the problem is open (e.g. for  $d = 6$ )
- if  $d = d_1 d_2$ , then  $N(d) \geq \min\{N(d_1), N(d_2)\}$
  
- W. Wootters and B.D. Fields
- S. Bandyopadhyay, P. Boykin, V. Roychowdhury, and F. Vatan
- A. Klappenecker and M. Rötteler
- M. Grassl
- T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski
- ...

## Three MUBs for $d = 2$

$$\begin{aligned}\mathcal{B}_1 &= \left\{ \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right\}, \\ \mathcal{B}_2 &= \left\{ \frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \frac{|0\rangle - i|1\rangle}{\sqrt{2}} \right\}, \\ \mathcal{B}_3 &= \{|0\rangle, |1\rangle\},\end{aligned}$$

eigenvectors of  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$

## MUBs – what are they good for ?

- quantum tomography
- quantum cryptography
- entropic uncertainty relations
- to witness entangled quantum states
- ...

## Generalized Pauli channel

$N(d) = d + 1$  maximal number of MUBs

$$\{ |\psi_0^{(\alpha)}\rangle, \dots, |\psi_{d-1}^{(\alpha)}\rangle \} ; \quad \alpha = 1, 2, \dots, d + 1$$

$$P_l^{(\alpha)} = |\psi_l^{(\alpha)}\rangle\langle\psi_l^{(\alpha)}|$$

$$U_\alpha = \sum_{l=0}^{d-1} \omega^l P_l^{(\alpha)} ; \quad (\omega = e^{2\pi i/d})$$

$$\mathbb{U}_\alpha[\rho] = \sum_{k=1}^{d-1} U_\alpha^k \rho U_\alpha^{k\dagger}$$

## Generalized Pauli channel

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## Generalized Pauli channel

$$\Phi = p_0 \mathbb{1} + \frac{1}{d-1} \sum_{\alpha=1}^{d+1} p_\alpha \mathbb{U}_\alpha$$

$$d = 2 \longrightarrow \Phi(\rho) = p_0 \rho + \sum_{\alpha=1}^3 p_\alpha \sigma_\alpha \rho \sigma_\alpha$$

- M. Nathanson and M.B Ruskai (2007) — *Pauli channels constant on axes*
- D. Petz and H. Ohno (2009)
- DC and K. Siudzińska (2016)

## Time-local master equation

$$\Lambda_t = p_0(t) \mathbb{1} + \frac{1}{d-1} \sum_{\alpha=1}^{d+1} p_\alpha(t) \mathbb{U}_\alpha$$

$$\frac{d}{dt} \Lambda_t = \mathcal{L}_t \Lambda_t$$

$$\mathcal{L}_t = \sum_{\alpha=1}^{d+1} \gamma_\alpha(t) (\Phi_\alpha - \mathbb{1})$$

$$\Phi_\alpha = \sum_{k=0}^{d-1} P_k^{(\alpha)} \rho P_k^{(\alpha)}$$

$\Phi_\alpha$  – quantum channel which perfectly decoheres w.r.t.  $|\psi_k^{(\alpha)}\rangle$

$$\mathcal{L}_t = \sum_{\alpha=1}^{d+1} \gamma_\alpha(t) (\Phi_\alpha - \mathbb{1})$$

### Theorem

- $\Lambda_t$  is invertible iff  $\Gamma_\alpha(t) = \int_0^t \gamma_\alpha(\tau) d\tau < \infty$ .
- invertible  $\Lambda_t$  is CPTP iff  $\gamma_\alpha(t) \geq 0$

### Theorem

- $\Lambda_t$  is P-divisible  $\longrightarrow \sum_{\beta \neq \alpha} \gamma_\beta(t) \geq 0$
- $\gamma_\alpha(t) + (d-1)\gamma_\beta(t) \geq 0 \longrightarrow \Lambda_t$  is P-divisible

$$\mathcal{L}_t = \sum_{\alpha=1}^{d+1} \gamma_\alpha(t) (\Phi_\alpha - \mathbb{1})$$

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- $\Lambda_t$  is P-divisible  $\rightarrow \sum_{\beta \neq \alpha} \gamma_\beta(t) \geq 0$
- $\gamma_\alpha(t) + (d-1)\gamma_\beta(t) \geq 0 \rightarrow \Lambda_t$  is P-divisible

## But what if the map is not invertible?

- B. Bylicka, M. Johansson, and A. Acín, Phys. Rev. Lett. **118**, 120501 (2017).
- DC, A. Rivas, and E. Størmer, arxiv 2017

## General $\Lambda_t$

$$\Lambda_t = V_{t,s} \circ \Lambda_s \quad ; \quad t \geq s$$

### Theorem

$\Lambda_t$  is divisible iff it is “kernel increasing”

$$\text{Ker}\Lambda_s \subset \text{Ker}\Lambda_t$$

$V_{t,s}$  is NOT uniquely defined on  $\mathcal{B}(\mathcal{H})$

$$V_{t,s} : \text{Im } \Lambda_s \rightarrow \text{Im } \Lambda_t$$

$$\text{Extension} : \longrightarrow \tilde{V}_{t,s} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$$

## General $\Lambda_t$

### Theorem

$$\text{If } \frac{d}{dt} \| [\mathbb{1} \otimes \Lambda_t](p_1 \rho_1 - p_2 \rho_2) \|_1 \leq 0$$

for all  $\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ , and  $p_1, p_2$ , then

- ①  $\Lambda_t$  is divisible
- ②  $V_{t,s}$  is CPTP on  $\text{Im } \Lambda_s$ .

Could we extend  $V_{t,s}$  to  $\mathcal{B}(\mathcal{H})$ ?

## Arveson extension theorem

$M \subset \mathcal{B}(\mathcal{H})$  – operator system

- $\mathbb{I} \in M$
- $X \in M \implies X^\dagger \in M$

### Theorem (Arveson)

Let  $M$  be an operator system and  $\Phi : M \rightarrow \mathcal{B}(\mathcal{H})$  a unital CP map. Then there exists unital CP extension  $\tilde{\Phi} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ .

It is NOT true for positive maps !!!

## Generalizing Arveson theorem

$M \subset \mathcal{B}(\mathcal{H})$  – spanned by positive operators

Theorem (Jencova)

If  $\Phi : M \rightarrow \mathcal{B}(\mathcal{H})$  is a CP map, then there exists CP extension  
 $\tilde{\Phi} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ .

It is NOT true for positive maps !!!

### Theorem (DC,Rivas,Størmer)

$$\text{If } \frac{d}{dt} \| [\mathbb{1} \otimes \Lambda_t](p_1 \rho_1 - p_2 \rho_2) \|_1 \leq 0$$

for all  $\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ , and  $p_1, p_2$ , then  $V_{t,s}$  is CPTP on  $\text{Im } \Lambda_s$ , and it can be extended to CP map

$$\tilde{V}_{t,s} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$$

$\tilde{V}_{t,s}$  is always trace-preserving on  $\text{Im } \Lambda_s$

$\tilde{V}_{t,s}$  needs NOT be trace-preserving !!!

**Could we have both CP and trace-preservation ?**

$$\Lambda_t = V_{t,s} \circ \Lambda_s \quad ; \quad t \geq s$$

$\Lambda_t$  is “image decreasing”  $\iff \text{Im}\Lambda_t \subset \text{Im}\Lambda_s$

### Theorem (DC,Rivas,Størmer)

If  $\Lambda_t$  is image decreasing, then it is CP-divisible iff

$$\frac{d}{dt} \| [\mathbb{1} \otimes \Lambda_t](p_1 \rho_1 - p_2 \rho_2) \|_1 \leq 0$$

for all  $\rho_1, \rho_2 \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ , and  $p_1, p_2$ ,

## Example – image decreasing

- $\Lambda_{t_1} \circ \Lambda_{t_2} = \Lambda_{t_2} \circ \Lambda_{t_1}$
- $\Lambda_t$  is diagonalizable

### Theorem

*If  $\Lambda_t$  is kernel increasing, then it is image decreasing.*

Majority of studied examples fit this class

## Example – amplitude damping

$$\Lambda_t(\rho) = \begin{pmatrix} |G(t)|^2 \rho_{11} & G(t) \rho_{12} \\ G^*(t) \rho_{21} & (1 - |G(t)|^2) \rho_{11} + \rho_{22} \end{pmatrix}$$

$$G(0) = 1 ; \quad |G(t)| \leq 1$$

- $\Lambda_t$  is invertible iff  $G(0) \neq 0$
- if  $G(t_*) = 0$ , then  $\Lambda_t$  is kernel increasing iff  $G(t) = 0$  for  $t > t_*$
- kernel increasing  $\implies$  image decreasing
- $\Lambda_t$  is CP-divisible iff  $\frac{d}{dt}|G(t)| \leq 0$  for  $t \leq t_*$ .

## Example – amplitude damping

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$$L_t(\rho) = \frac{-is(t)}{2} [\sigma_+ \sigma_-, \rho] + \gamma(t) ([\sigma_-, \rho \sigma_+] + [\sigma_- \rho, \sigma_+])$$

$$s(t) = -2 \operatorname{Im} \frac{\dot{G}(t)}{G(t)} ; \quad \gamma(t) = -2 \operatorname{Re} \frac{\dot{G}(t)}{G(t)}$$

Markovianity  $\longleftrightarrow \gamma(t) \geq 0$  for  $t < t_*$

but  $\gamma(t)$  might be negative for  $t > t_*$  !!!

# Conclusions

- Markovianity = CP-divisiblity  $\longleftrightarrow \Lambda_t = V_{t,s} \circ \Lambda_s$
- for invertible maps: CP-divisibility  $\longleftrightarrow$  monotonicity of the trace norm
- we generalize it for non-invertible maps **but**
- for non-invertible maps:  $V_{t,s}$  might be CP on  $\mathcal{B}(\mathcal{H})$  but trace-preserving only on  $\text{Im}\Lambda_s$
- for image decreasing maps it is also trace-preserving on  $\mathcal{B}(\mathcal{H})$
- as a byproduct we generalized the structure of time-dependent Lindblad master equation