# Multi-symplectic Lie group thermodynamics for gauge theories: a work in progress

Goffredo Chirco

joint work with Marco Laudato & Fabio M. Mele

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# Motivation

In a general covariant theory, as GR, there is no preferred time flow, and the dynamics of the theory cannot be formulated in terms of an evolution in a single external time parameter. What is left of statistical mechanics?

the problem of a generalized covariant framework for statistical mechanics comes with a rich set of theoretical open problems directly related to our understanding of gravity and quantum gravity:

- consistently describe gravitational field fluctuations; account for the evidence of puzzling thermodynamic aspects of gravity: horizon thermodynamics, emergent gravity, Einstein's equations as equation of state...
- preferred time in stat mech: conceptual incompatibility lying at the root of the quantum gravity puzzle
- general interest: such a framework of open questions brings together theoretical aspects of (quantum) information theory, statistical mechanics, gravity and quantum mechanics

=> room for new principles in (quantum) gravity from information theory? (for instance in gravity, holography, duality are still conjectured)

## Outline: take a geometric point of view

- STAT MECH on SYMPLECTIC MANYFOLD
  - geometric description of standard Gibbs state
  - Hamiltonian momentum map

- GENERALIZATION TO LIE GROUP ACTIONS (Souriau)
  - Generalized Gibbs state w.r.t. action of a Lie group

- GENERALIZATION TO GAUGE GROUP ACTIONS
  - multisymplectic formalism for covariant field theory
  - generalized Gibbs state from covariant momentum map
  - reduction to instantaneous framework

# non-rel. Hamiltonian mechanics

- Let $N$ be the configuration manifold of a Lagrangian system whose Lagrangian $L : TN \to \mathbb{R}$ is hyper-regular and does not explicitly depend on the time $t$. Let $H : T^*N \to \mathbb{R}$ be the corresponding Hamiltonian and $(M, \omega)$ be the symplectic (maybe non-Hausdorff) manifold of motions, namely the reduced phase space of the system.
- the Hamiltonian $H : T^*N \to \mathbb{R}$ remains constant along each motion of the system. Therefore it is possible to define on the symplectic manifold of motions $(M, \omega)$ a smooth function $E : M \to \mathbb{R}$, called the *energy function* (derived from the lagrangian...)

$$E(\varphi) = H(\varphi(t)) \quad \text{for all } t \in R, \quad \varphi \in M.$$

- the Hamiltonian vector field $X_E$ on $M$ is the infinitesimal generator of the 1-dimensional group of time translations. A time translation $\Delta t : \mathbb{R} \to \mathbb{R}$ is a map $\Delta t : \mathbb{R} \to \mathbb{R}$, $\Delta t(t) = t + \Delta t$. The group of time translations can be identified with $\mathbb{R}$. It acts on the manifold of motions $M$ by the action $\Phi^E$ , such that for each time translation $\Delta t$ and each motion $\varphi$, $\Phi^E_{\Delta t}(\varphi)$ is the motion

$$t \to \Phi^E_{\Delta t}(\varphi(t)) = \varphi(t + \Delta t).$$

## Statistical mechanics on a symplectic manifold

- A *statistical state* on $(M, \omega)$ is a probability measure $\rho$ on the symplectic manifold $M$ defined by

$$\rho(A) = \int_A \rho(\varphi)\,\omega^n(\varphi)$$

for each Borel subset $A$ of $M$, with $\rho : M \to \mathbb{R}\,([0, +\infty[)$ being a continuous density function wrt the natural volume form $\omega^n$ (Liouville measure) on $M$, with $n = \dim(M)$, such that $\int_M \rho(\varphi)\,\omega^n(\varphi) = 1$.

- To such state, one can associate an entropy:

$$s(\rho) = -\int_M \rho(\varphi)\,\log\left(\rho(\varphi)\right)\omega^n(\varphi)$$

with the convention that if $x \in M$ is such that $\varphi(x) = 0$, $\log(\varphi(x))\varphi(x) = 0$.

- For each $f$ on $M$, taking its values in $\mathbb{R}$ or in some finite-dimensional vector space, such that the integral on the right hand side of the equality

$$\mathcal{E}_\rho(f) = \int_M f\,\rho\,\omega^n$$

converges, the value $\mathcal{E}_\rho(f)$ of that integral is called the mean value of $f$ with respect to $\rho$.

# Statistical mechanics on a symplectic manifold

- a thermodynamic equilibrium state can be defined via entropy maximization (Jaynes), as a statistical state with a smooth probability density $\rho \geq 0$, satisfying the two constraints

$$\int_M \rho(\varphi)\, \omega^n(\varphi) = 1$$

$$\int_M \rho(\varphi)\, E(\varphi)\, \omega^n(\varphi) = Q$$

  and such that the entropy function $s(\rho)$ is stationary with respect to all inf. smooth variations of the probability density,for a given value mean value $Q$ of the energy function $E$.

- in particular, the entropy function $s$ is stationary iff there exists a real $B \in \mathbb{R}$ such that, $\forall \varphi \in M$,

$$\rho_B(\varphi) = \frac{1}{P(B)} e^{-BE(\varphi)}$$

  called the Gibbs statistical state associated to $B$, with $P(B)$ the partition function

$$P(B) = \int_M e^{-B\,E(\varphi)}\, \omega^n(\varphi).$$

# Statistical mechanics on a symplectic manifold

- For $H$ smooth Hamiltonian bounded by below, the Gibbs state associated to $B$ is invariant under the flow of of the Hamiltonian vector field $X_H$.
- At equilibrium, one can define $\Psi$ and $Q$ smooth functions of $B$,

$$\begin{aligned} Q(B) &= \frac{1}{P}\int_M E(\varphi)\, e^{-B\, E(\varphi)}\, \omega^n(\varphi), \\ \Psi(B) &= -\log P(B). \end{aligned}$$

  In particular, by means of convexity arguments, it follows that when $Q$ is given, there is at most one corresponding value of $B$, so that $\Psi(B)$ and the probability density $\rho$ are uniquely determined. The value of $s(\rho)$ at equilibrium, given by

$$s(B) = \Psi(B) + B\, Q(B).$$

- this allows us to interpret $Q$ as the internal energy of the system, while $\Psi$ as the free energy, with $B$ playing the role of the inverse temperature.

## Generalization for a Hamiltonian Lie group action

The energy function $E$ on $(M, \omega)$ can be seen as the momentum map of the Hamiltonian action $\Phi^E$ on that manifold of the one-dimensional Lie group of time translations.

More generally:

- *Hamiltonian momentum map*: Let $\psi$ be a Hamiltonian action of a finite-dimensional Lie algebra $\mathfrak{g}$ on a symplectic manifold $(M, \omega)$. There exists a smooth map $J : M \to \mathfrak{g}^*$, taking its values in the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, such that for each $Y \in \mathfrak{g}$ the Hamiltonian vector field $\psi(Y)$ on $M$ admits as Hamiltonian the function $J_Y : M \to \mathbb{R}$, defined by

$$J_Y(x) = \langle J(x), Y \rangle, x \in M.$$

=> follows a natural generalization of the definition of a thermodynamic equilibrium state in which a Lie group $G$ acts, by a Hamiltonian action $\Phi$, on that symplectic manifold. (Souriau)

# Generalized Gibbs state (Marle)

Consider then a connected symplectic manifold $(M, \omega)$ and with a connected Lie group $G$ acting on $M$ by a Hamiltonian action $\Phi$. Let $\mathfrak{g}$ be the Lie algebra of $G$, $\mathfrak{g}^*$ be its dual space and $J : M \to \mathfrak{g}^*$ a *momentum map* of the action $\Phi$. A generalized temperature is an element $B \in \mathfrak{g}$ such that the integral

$$\int_M e^{-\langle J(\varphi), B\rangle} \omega^n,$$

*normally* converges. A generalized Gibbs state associated to $B \in \Omega \subset \mathfrak{g}$, is the statistical state defined by probability density

$$\rho_B = \frac{1}{P(B)} e^{-\langle J(\varphi), B\rangle},$$

with respect to the natural volume form on the simplectic manifold $M$. The quantity

$$P(B) = \int_M e^{-\langle J(\varphi), B\rangle} \omega^n,$$

defines the generalized partition function. The Hamiltonian function $\langle J(\varphi), B\rangle : M \to \mathbb{R}$ is the *comomentum map* defined by the natural pointwise pairing.

# Important remarks

- Differently from the case of time translations ($G = \mathbb{R}$), a generalized Gibbs state of a Lie group $G$ does not necessarily define thermodynamic equilibrium state with respect to the generic Hamiltonian action $\Phi_G$, as it may not be invariant with respect to the action of the Lie group $G$ on the symplectic manifold of motions $(M, \omega)$.
- indeed the probability density $\rho_B$ involves the value of the momentum map $J$, which is *equivariant* with respect to the action $\Phi$ of $G$ on $M$ and an affine action of $G$ on the dual of its Lie algebra $\mathfrak{g}^*$

  $$J \circ \Phi_g = Ad^*_{g^{-1}} \circ J + \theta(g)$$

  where $\theta : G \to \mathfrak{g}^*$ is the symplectic cocycle of G for the coadjoin action of $G$ on $\mathfrak{g}^*$, for any $g \in G$
- In particular, the generalized Gibbs state associated to $B$ is only invariant under the *restriction* of the Hamiltonian action $\Phi$ to the *one-parameter subgroup* of $G$ generated by $B$, $\{\exp(\tau B) | \tau \in \mathbb{R}\}$.

if a generalized Gibbs states can be defined, due to the infinite differentiability of the generalized partition function, generalized thermodynamic relations will follow from the definition of generalized macroscopic quantities in terms of differentials of $Z$.

- the internal energy is given by the average momentum map within the canonical ensemble, from the first differential of $\Psi : \mathfrak{g} \to \mathbb{R}$,

$$Q(B) = -D\Psi(B) = -D(\log Z(B))$$

- as for an equilibrium state, when $Q$ is given, there is at most one corresponding value of $B$, so that $\Psi(B)$ and the probability density $\rho$ are uniquely determined. Accordingly, one can prove that the entropy function $s(\rho)$ has a strict maximum $S(B)$, with respect to smooth variations of $\rho$ satisfying the constraints, so that

$$\begin{aligned} S(B) &= \Psi(B) + B\,Q(B) \\ &= \log Z(B) - \langle D(\log Z(B)), B\rangle \end{aligned}$$

- the first differential of the entropy, for all $B \in \Omega$ can be considered as an element of the $\mathfrak{g}^*$, defined for $Y \in \mathfrak{g}$ by

$$\begin{aligned} \langle DS(B), Y \rangle &= \langle DQ_J(B)(Y), B \rangle \\ &= -\langle D(D\Psi(B))(Y), B \rangle < 0. \end{aligned}$$

- for any $Y \in \mathfrak{g} \setminus \{0\}$, the second differential $D^2\Psi(B)$ defines a positive symmetric bilinear form, which provides a generalized notion of heat capacity.
- considering equations of state to pass from thermostatics to actual thermodynamics!

Souriau's key point: defining a statistical state on the space of solutions allows to describe thermodynamics in a covariant way, without fixing preferred frames which reduce the full symmetry of the system

=> can we use a similar setting for general covariant (relativistic) field theory and ultimately for gravity?

# Generalization

In general, we would need

- covariant finite dimensional space for formulating hamiltonian mechanics for fields
- multisymplectic phase space construction
- covariant multi-momentum map

# Ways to proceed

Two ways to proceed:

1. reduce to physical space of covariant field theory and play with reduced momentum maps associated to actual conserved charges
2. work at the level of multisymplectic space and use the generalized Gibbs as an unphysical equilibrium state which defines the very partition function of the theory. Hence try to extract dynamics from the resultant thermodynamics

# Generalization to multisymplectic formalism (Gotay Marsden)

- Let $\mathcal{X}$ be an oriented $n+1$-dimensional manifold, which in many examples is spacetime, and let $\mathcal{Y} \xrightarrow{\pi_{\mathcal{X}\mathcal{Y}}} \mathcal{X}$ be a finite-dimensional fiber bundle over $\mathcal{X}$ whose fibers $\mathcal{Y}_x$ over $x \in \mathcal{X}$ have dimension $N$. This is called the covariant configuration bundle and is the covariant analogue of the configuration space in classical mechanics.
- Physical fields correspond to sections of this bundle. A set of local coordinates $(x^\mu, y^A)$ on $\mathcal{Y}$ is provided by the $n+1$ local coordinates $x^\mu$, $\mu = 0, \ldots, n$, on $\mathcal{X}$ and the $N$ fiber coordinates $y^A$, $A = 1, \ldots, N$, which represent the field components at a given point $x \in \mathcal{X}$.
- Such notion of extended configuration space for field theories has a nice operational motivation based on the observation that coordinates of $\mathcal{Y}$ (i.e., field values and spacetime positions) are the partial observables of the theory (Rovelli). Indeed, one needs $N$ measuring devices to measure the components of the field at a given point $x \in \mathcal{X}$, and $n+1$ devices to determine $x$ thus resulting in a $(n+N+1)$-dimensional configuration space.

# Tangent bundle

- A point in $\mathcal{Y}$ represents a correlation between these observables, that is, a possible outcome of a simultaneous measurements of the partial observables.
- the equivalence class of all (local) sections s at $x$ which have the same first order Taylor expansion is called the 1-jet prolongation of $s$ at $x$ and is denoted by $j^1(s)_x$. The resulting set of equivalence classes

  $$J^1(\mathcal{Y}) = \left\{ j^1(s)_x \mid x \in U \subset \mathcal{X} \, , \, s \in \Gamma(U, \mathcal{Y}) \right\}$$

  is called the first jet bundle of $\mathcal{Y}$. If $s : \mathcal{X} \to \mathcal{Y}$ is a section of $\pi_{\mathcal{X}\mathcal{Y}}$, its tangent map $T_x s$ at $x \in \mathcal{X}$ is an element of $J^1_{s(x)}(\mathcal{Y})$. Thus, the map $j^1(s) : x \to T_x s$ is a section of $J^1(\mathcal{Y})$ regarded as a bundle over $\mathcal{X}$

  $$\begin{array}{ccc} J^1(\mathcal{Y}) & \xrightarrow{\pi^1_{\mathcal{Y}}} & \mathcal{Y} \\ \downarrow{\scriptstyle \pi^1_{\mathcal{X}}} & & \downarrow{\scriptstyle \pi_{\mathcal{X}\mathcal{Y}}} \\ \mathcal{X} & \xrightarrow{\mathrm{id}_{\mathcal{X}}} & \mathcal{X} \end{array}$$

  where $\pi^1_{\mathcal{X}}(j^1(s)_x) = x$, and $\pi^1_{\mathcal{Y}}(j^1(s)_x) = j^0(s)_x \equiv s(x) = (x^\mu, y^A)$.
- In the context of field theories, the first jet bundle $J^1(\mathcal{Y})$ of $\mathcal{Y}$ plays the role of the tangent bundle of classical mechanics

# Multi Phase Space or Covariant Phase Space

- the field-theoretic analogue of the cotangent bundle is given by the dual jet bundle $J^1(\mathcal{Y})^*$ defined as the vector bundle over $\mathcal{Y}$ whose fiber at $y \in \mathcal{Y}_x$ is the set of affine maps from $J^1_y(\mathcal{Y})$ to $\Lambda^{n+1}_x(\mathcal{X})$, where $\Lambda^{n+1}(\mathcal{X})$ denotes the bundle of $(n+1)$-forms on $\mathcal{X}$.
- an equivalent and more convenient description of $J^1(\mathcal{Y})^*$ can be given as follows: Let $\Lambda := \Lambda^{n+1}(\mathcal{Y})$ be the bundle of $(n+1)$-forms on $\mathcal{Y}$ with fiber $\Lambda_y$ over $y \in \mathcal{Y}$ and projection $\pi_{\mathcal{Y}\Lambda} : \Lambda \to \mathcal{Y}$. Let $\mathcal{Z} \subset \Lambda$ be the subbundle of 2-horizontal $(n+1)$-forms on $\mathcal{Y}$, that is the bundle whose fiber over $y \in \mathcal{Y}$ is given by (De Donder Weyl)

$$\mathcal{Z}_y = \{ z \in \Lambda_y \,|\, \mathrm{i}_V \mathrm{i}_W z = 0 \quad \forall\, V, W \in T_y\mathcal{Y} \ \text{ s.t. } \ T\pi_{\mathcal{X}\mathcal{Y}} \cdot V = 0,\ T\pi_{\mathcal{X}\mathcal{Y}} \cdot W = 0 \} \ ,$$

  where $\mathrm{i}_V$ denotes left interior multiplication by $V$. The elements of $\mathcal{Z}$ can be be written uniquely as $z = p\,\mathrm{d}^{n+1}x + p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^n x_\mu$
- therefore, in complete analogy to classical mechanics, we define the canonical Poincaré-Cartan $(n+1)$-form $\Theta$ on $\mathcal{Z}$

$$\Theta = p\,\mathrm{d}^{n+1}x + p^\mu_A \mathrm{d}y^A \wedge \mathrm{d}^n x_\mu \ .$$

- the canonical $(n+2)$-form $\Omega$ on $\mathcal{Z}$ is defined by $\Omega = -\mathrm{d}\Theta$ whose coordinate expression reads

$$\Omega = \mathrm{d}y^A \wedge \mathrm{d}p^\mu_A \wedge \mathrm{d}^n x_\mu - \mathrm{d}p \wedge \mathrm{d}^{n+1}x.$$

summarizing the various spaces involved in the above constructions and their bundle relations:

$$
\begin{tikzcd}
(J^{1}(\mathcal{Y}), \Omega_{\mathcal{L}}) \arrow[r, "F\mathcal{L}"] \arrow[dr, "\pi^{1}_{\mathcal{Y}}"'] \arrow[ddr, "\pi^{1}_{\mathcal{X}}"'] & J^{1}(\mathcal{Y})^{*} \arrow[r, "\simeq"] \arrow[d, "\tilde{\pi}^{1}_{\mathcal{Y}}"'] \arrow[dd, bend left, "\tilde{\pi}^{1}_{\mathcal{X}}"] & (\mathcal{Z}, \Omega) \arrow[r, "i_{\Lambda\mathcal{Z}}"] \arrow[dl, "\pi_{\mathcal{Y}\mathcal{Z}}"] \arrow[ddl, "\pi_{\mathcal{X}\mathcal{Z}}"] & (\Lambda^{n+1}(\mathcal{Y}), \Omega_{\Lambda}) \arrow[dll, bend left, "\pi_{\mathcal{Y}\Lambda}"] \\
& \mathcal{Y} \arrow[d, "\pi_{\mathcal{X}\mathcal{Y}}"'] & & \\
& \mathcal{X} & &
\end{tikzcd}
$$

The pair $(\mathcal{Z}, \Omega)$ is called **multiphase space** or **covariant phase space**. It is an example of multisymplectic manifold defined as (GIMMSY,Helein) a manifold endowed with a closed nondegenerate $k$-form $\Omega$ ($k = n + 2$ in our case), i.e., such that $\mathrm{d}\Omega = 0$ and $\mathrm{i}_V \Omega \neq 0$ for any nonzero tangent vector $V$.

# Covariant Formalism for Field Theories

| **Classical Mechanics** ($n = 0, \mathcal{X} \equiv \mathbb{R}$) | **Field Theory** ($n > 0, \dim \mathcal{X} = n + 1$) |
|---|---|
| extended configuration space<br>$\mathcal{Y} = \mathbb{R} \times Q$ | configuration bundle over spacetime<br>$\mathcal{Y} \xrightarrow{\pi_{\mathcal{X}\mathcal{Y}}} \mathcal{X}$ |
| local coordinates on $\mathcal{Y}$<br>$(t, q^A)$ | local coordinates on $\mathcal{Y}$<br>$(x^\mu, y^A)$ |
| extended phase space<br>$\mathcal{P} = T^*\mathcal{Y} = T^*\mathbb{R} \times T^*Q$ | multiphase space<br>$J^1(\mathcal{Y})^* \cong \mathcal{Z} \subset \Lambda^{n+1}(\mathcal{Y})$ |
| local coordinates on $\mathcal{P}$<br>$(t, q^A, E, p_A)$ | local coordinates on $\mathcal{Z}$<br>$(x^\mu, y^A, p, p_A^\mu)$ |
| Poincaré-Cartan 1-form on $\mathcal{P}$<br>$\Theta = p_A \mathrm{d}q^A + E \mathrm{d}t$ | Poincaré-Cartan $(n{+}1)$-form on $\mathcal{Z}$<br>$\Theta = p \mathrm{d}^{n+1}x + p_A^\mu \mathrm{d}y^A \wedge \mathrm{d}^n x_\mu$ |
| symplectic 2-form on $\mathcal{P}$<br>$\Omega = \mathrm{d}q^A \wedge \mathrm{d}p_A - \mathrm{d}E \wedge \mathrm{d}t$ | multisymplectic $(n{+}2)$-form on $\mathcal{Z}$<br>$\Omega = \mathrm{d}y^A \wedge \mathrm{d}p_A^\mu \wedge \mathrm{d}^n x_\mu - \mathrm{d}p \wedge \mathrm{d}^{n+1}x$ |

# Generalized Gibbs state on multiphase space

The goal is now to extend Souriau's definition of generalized Gibbs state to the multisymplectic framework, with respect to the action of diffeos.

Premises

- Let $\mathcal{G}$ be a Lie group (perhaps infinite-dimensional) with Lie algebra $\mathfrak{g}$ that acts on $\mathcal{X}$ by diffeomorphisms and acts on $\mathcal{Z}$ (or $\mathcal{Y}$ ) as $\pi_{\mathcal{X}\mathcal{Z}}$- (or $\pi_{\mathcal{X}\mathcal{Y}}$-) bundle automorphisms.
- Given an element $\xi \in \mathfrak{g}$, we denote by $\xi_{\mathcal{X}}, \xi_{\mathcal{Y}}$, and $\xi_{\mathcal{Z}}$ the infinitesimal generators of the corresponding transformations on $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$, i.e., the infinitesimal generators on $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ of the one-parameter group generated by $\xi$. The group $\mathcal{G}$ is said to act on $\mathcal{Z}$ by *covariant canonical transformation* if this action corresponds to an infinitesimal multisymplectomorphism, i.e.

  $$\mathcal{L}_{\xi_{\mathcal{Z}}}\Omega = 0 \ ,$$

  where $\mathcal{L}_{\xi_{\mathcal{Z}}}$ denotes the Lie derivative along $\xi_{\mathcal{Z}}$, while it is said to act by *special covariant canonical transformations* if

  $$\mathcal{L}_{\xi_Z}\Theta = 0 \ .$$

# Covariant Multimomentum map

In analogy to the definition of momentum maps in symplectic geometry, a covariant momentum map (or a **multimomentum map**) for the action of $\mathcal{G}$ on $\mathcal{Z}$ by covariant canonical transformations is a map

$$J : \mathcal{Z} \longrightarrow \mathfrak{g}^* \otimes \Lambda^n(\mathcal{Z}) \ ,$$

covering the identity on $\mathcal{Z}$ such that

$$\mathrm{d}J(\xi) = \mathrm{i}_{\xi_{\mathcal{Z}}}\Omega,$$

where $J(\xi)$ is the $n$-form on $\mathcal{Z}$ whose value at $z \in \mathcal{Z}$ is $\langle J(z), \xi \rangle$ with $\langle \cdot, \cdot \rangle$ being the pairing between the Lie algebra $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. Let $\phi_\xi \in \mathcal{G}$ be the transformation associated to $\xi \in \mathfrak{g}$, then a covariant momentum map is said to be $\mathrm{Ad}^*$-equivariant if

$$J(\mathrm{Ad}^{-1}\xi) = \phi_\xi^*[J(\xi)] \ .$$

If $\mathcal{G}$ acts on $\mathcal{Z}$ by special covariant canonical transformations, the special covariant momentum map admits an explicit expression given by

$$J(\xi) = \mathrm{i}_{\xi_{\mathcal{Z}}}\Theta \ ,$$

so that $\mathrm{d}J(\xi) = \mathrm{d}\mathrm{i}_{\xi_{\mathcal{Z}}}\Theta = (\mathcal{L}_{\xi_{\mathcal{Z}}} - \mathrm{i}_{\xi_{\mathcal{Z}}}\mathrm{d})\Theta = \mathrm{i}_{\xi_{\mathcal{Z}}}\Omega$. In particular, if the action of $\mathcal{G}$ on $\mathcal{Z}$ is the lift of an action of $\mathcal{G}$ on $\mathcal{Y}$ , then $\xi_{\mathcal{Y}} = T\pi_{\mathcal{Y}\mathcal{Z}} \cdot \xi_{\mathcal{Z}}$ and, according to the definitions of $\Theta$ and $J(\xi)$, the **special covariant momentum map** is given by

$$J(\xi)(z) = \pi^*_{\mathcal{Y}\mathcal{Z}}\mathrm{i}_{\xi_{\mathcal{Y}}} z \ .$$

Denoting by $(\xi^\mu, \xi^A)$ the components of $\xi_{\mathcal{Y}}$. In coordinates

$$J(\xi)(z) = (p^\mu_A \xi^A + p\,\xi^\mu)\mathrm{d}^n x_\mu - p^\mu_A \xi_\nu \mathrm{d}y^A \wedge \mathrm{d}^{n-1} x_{\mu\nu},$$

where $\mathrm{d}^{n-1}x_{\mu\nu} = \mathrm{i}_{\partial_\nu}\mathrm{i}_{\partial_\mu}\mathrm{d}^{n+1}x$.

## Covariant Multimomentum map

our previous setting was trying a straightforward generalisation of the symplectic derivation by considering

$$
\begin{array}{ccc}
\text{symplectic} & & \text{multisymplectic} \\
J : M \to \mathfrak{g}^* & \longrightarrow & J : \mathcal{M} \longrightarrow \mathfrak{g}^* \otimes \Lambda^{n-1} T^*\mathcal{M} \\
\overline{J} = \int_M \mathrm{vol}(M)\, \rho_M\, J & & \overline{J} = \underbrace{\int_{\mathcal{M}} \mathrm{vol}(\mathcal{M})\, \rho_{\mathcal{M}}\, J}_{\substack{\text{not possible} \\ \text{as the integral} \\ \text{is saturated by} \\ \text{the volume form}}}
\end{array}
\tag{1}
$$

How do we move forward? Use the notion of observable Hamiltonian (n-1)-form for the multimomentum map.

## Covariant thermodynamic equilibrium state

**1)** Covariant momentum map $J : \mathcal{M} \longrightarrow \mathfrak{g}^* \otimes \Lambda^{n-1} T^* \mathcal{M}$ such that

$$dJ(\xi) = \xi_z \lrcorner \, \omega$$

where $\xi \in \mathfrak{g}$ and $J(\xi)$ is the $n-1$ form on $\mathcal{M}$ whose value at $z \in \mathcal{M}$ is $\langle J(\xi), \xi \rangle$.
$\Rightarrow \quad J(\xi)$ is an **observable** $n-1$ form (Hélein&Kouneiher).

**2)** Generally, observable $n-1$ forms, say $F$, can be integrated over co-dim one hyper-surfaces in an $n$-curve to produce observable functionals. Given some Hamiltonian function on $\mathcal{M}$, define a slice $\Sigma$ to be a co-dim one submanifold of $\mathcal{M}$ such that for any Hamiltonian $n$-curve $\Gamma$, the intersection $\Sigma \cap \Gamma$ is transverse (assume $\Sigma$ oriented $\rightarrow$ endow $\Sigma \cap \Gamma$ with orientation). Hence, we have

$$\int_\Sigma F \; : \mathcal{F} \longrightarrow \mathbb{R}$$
$$\Gamma \longmapsto \int_{\Sigma \cap \Gamma} F$$

where $\mathcal{F}$ is the set of $n$-dimensional oriented submanifolds of $\mathcal{M}$ (space of paths).

**3)** Analogously for the $\mathfrak{g}^*$-valued $n-1$ form $J$, we define the observable momentum map functional as

$$j[\Gamma] := \int_\Sigma J \; : \mathcal{F} \longrightarrow \mathfrak{g}^*$$
$$\Gamma \longmapsto \int_{\Sigma\cap\Gamma} J$$

**4)** **<u>mean value</u>**
– as a functional on $\mathcal{F} = \{\Gamma\}$, the mean value of $j$ should be defined on $\mathcal{F}$ as

$$\bar{j} = \int_{\mathcal{F}} \mathrm{vol}(\mathcal{F})\, \rho_{\mathcal{F}}\, j$$

where $\rho_{\mathcal{F}}$ is a probability density on $\mathcal{F}$ such that $\mu(\mathcal{A}) = \int_{\mathcal{A}\subset\mathcal{F}} \mathrm{vol}(\mathcal{F})\, \rho_{\mathcal{F}}$.

if going on the path space is correct, then also the variational derivation of the Gibbs state should be carried over there!

**5)** **variation of the entropy**:

$$\begin{cases} S = -\int_{\mathcal{F}} \mathrm{vol}(\mathcal{F})\, \rho_{\mathcal{F}} \log \rho_{\mathcal{F}} \\ \text{constraint } \bar{j} = \text{const.}\ ,\ \beta \text{ Lagrange multiplier vector in } \mathfrak{g} \end{cases}$$

$$\begin{aligned} D_\epsilon S &= -\int_{\mathcal{F}} \mathrm{vol}(\mathcal{F})\ (1+\log\rho_{\mathcal{F}})\, D_\epsilon \rho_{\mathcal{F}} - D_\epsilon \left\langle \beta, \bar{j} \right\rangle \\ &= -\int_{\mathcal{F}} \mathrm{vol}(\mathcal{F})\ (1+\log\rho_{\mathcal{F}})\, D_\epsilon \rho_{\mathcal{F}} - \left\langle \beta, D_\epsilon \left[ \int_{\mathcal{F}} \mathrm{vol}(\mathcal{F})\, \rho_{\mathcal{F}}\, j \right] \right\rangle \\ &= -\int_{\mathcal{F}} \mathrm{vol}(\mathcal{F})\ (1+\log\rho_{\mathcal{F}})\, D_\epsilon \rho_{\mathcal{F}} - \int_{\mathcal{F}} \mathrm{vol}(\mathcal{F})\, D_\epsilon \rho_{\mathcal{F}}\ \langle \beta, j \rangle \\ &= -\int_{\mathcal{F}} \mathrm{vol}(\mathcal{F})\, D_\epsilon \rho_{\mathcal{F}} \left( 1 + \log\rho_{\mathcal{F}} + \left\langle \beta, \int_{\Sigma\cap\Gamma} J \right\rangle \right) = 0 \qquad \forall\ D_\epsilon \rho_{\mathcal{F}}\ , \end{aligned}$$

i.e.

$$\rho_{\mathcal{F}}^{(eq)}(\Gamma, \beta) \propto e^{-\left\langle \beta, \int_{\Sigma\cap\Gamma} J \right\rangle} \equiv e^{-\mathcal{G}_\beta[\Gamma]}$$

our Gibbs state is now a functional of the $n$-curve $\Gamma$.

# Laplace transform

We can use *generalised* ensemble theory to find a relation among generalized Gibbs state and the usual delta-like meausre on the multi-presymplectic space. The canonical partition function is the Laplace transform of the microcanonical partition function $\Omega(N, V, E)$. We can invert this relation by applying the inverse Laplace transform to $Q$:

$$\Omega(N, V, E) = \frac{1}{2\pi i} \oint_{-\infty}^{\infty} \mathrm{d}\beta \, e^{\beta E} Q(N, V, \beta) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} \mathrm{d}\beta \, e^{\beta(E-H)}$$

where the phase space differential form $\mathrm{d}\Omega = (h^{3N} N!)^{-1} \mathrm{d}x_1, \ldots, \mathrm{d}x_{3N}, \mathrm{d}p_1, \ldots \mathrm{d}p_{3N}$ . Now, $\beta = \sigma + i\tau$ and because no singularity is present in the right-half of the complex plane, the contour may be taken vertically through $\gamma = 0$. Since $Re(\beta) = 0$ along the integration, the substitution $\beta = -i\tau$ can be made and we have,

$$\Omega(N, V, E) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\Omega}{2\pi} \int_{\infty}^{\infty} \mathrm{d}\tau \, e^{i\tau(E-H)} = \int_{-\infty}^{\infty} \mathrm{d}\Omega \, \delta(H - E).$$

which can be identified as the microcanonical partition function.

# Contact with the instantaneous framework (Gimmsy)

We need to make contact with the base manifold:

- n-dimensional $\Gamma$ submanifolds of $\mathcal{M}$ can be equivalently thought of as a sections $s : \mathcal{U} \subset \mathcal{X} \to \mathcal{M}$, where $s : x \mapsto s(x) = (x, \phi(x), p_{\phi(x)})$.
- let $\Sigma$ be a compact (oriented, connected) boundaryless $(n-1)$-manifold. We denote by $Emb(\Sigma, \mathcal{X})$ the space of all smooth embeddings of $\Sigma$ into $\mathcal{X}$.
- then we expect we can associate

  $$\int_{\Sigma \cap \Gamma} \langle \beta, J \rangle \longleftrightarrow \int_{\Sigma_\tau} s^* \langle \beta, J \rangle \equiv \langle \beta, E_\tau(s) \rangle$$

  where $E_\tau : \mathcal{Z}_\tau = \Gamma(\Sigma_\tau \subset X, \mathcal{Z}) \to \mathfrak{g}^*$ (Energy-Momentum map)
- denote $T^*\mathcal{Y}_\tau = \mathcal{Z}_\tau / Ker\Omega_\tau$ the instantaneous symplectic space
- denote $\mathcal{P}_\tau \subset T^*\mathcal{Y}_\tau$ the instantaneous presymplectic space or $\tau$-primary constraint set
- $E_\tau$ projects to $\mathcal{E}_\tau : \mathcal{P}_\tau \to \mathfrak{g}^*$ (*Instantaneous* Energy-Momentum map)

Remarks

$$\langle \beta, \mathcal{E}_\tau(s) \rangle = \int_{\Sigma_\tau} \langle \beta, \mathcal{E}_\tau(\phi, \pi) \rangle \,, \quad (\phi, \pi) \in \mathcal{P}_\tau$$

$$\begin{cases} \text{if} \beta_X \text{ (for } \beta \in \mathfrak{g}) \text{ is transverse to } \Sigma_\tau, \text{ then } \langle \beta, \mathcal{E}_\tau(s) \rangle = -H_{\tau,\beta}(\phi, \pi) \\ \text{if} \beta_X \text{is tangent, then } \langle \beta, \mathcal{E}_\tau(s) \rangle = \langle \beta, j_\tau(\phi, \pi) \rangle \end{cases}$$

- where $j_\tau : T^*\mathcal{Y}_\tau \to \mathfrak{g}^*$ is the momentum map for the induced action of $\mathcal{G}_\tau$ on $T^*\mathcal{Y}_\tau$. Notice that $\mathcal{G}_\tau$s stabilize the image of $\tau$ (Cauchy surfaces).
- vanishing of the components of $\mathcal{E}_\tau$ are the first class secondary constraint functions of the system.

=> we expect to be able to define the physical partition function of the theory starting from the instantaneous framework...

# Conclusions

- we realized a gauge group thermodynamic framework based on the notion of covariant momentum map, leading to the definition of a generalized Gibbs state for the action of lifted automorphisms of the bundle covering diffeos on the base.

=> explore thermodynamics (role of boundaries and cocycles?)

- A key feature of relativistic field theories is that not all of the Euler-Lagrange equations necessarily describe the temporal evolution of fields. Some of the equations may impose constraints on the choice of initial data. Those constraints which are first class in the sense of Dirac reflect the gauge symmetry of the theory. These first class constraints are related to the vanishing of various momentum maps (GIMMSY).

=> may exploit generalized Gibbs state as a generalization of the presymplectic formalism ( "statistical" symplectic reduction ?), while having a consistent thermodynamical relation withthe standard presymplectic formalism (via Laplace).

$$J \approx 0 \to \bar{J} = -D\Psi(\beta) \approx 0$$

# Perspectives

- understand statistical symplectic reduction: viable/meaningful in first place? What is the meaning of an averaged constraint?
- contact with the quantum mechanics: the fact that a state defines a one parameter family of automorphisms is a fundamental property of von Neumann algebras. The relation between a state over an algebra and a one parameter family of automorphisms of the algebra is the content of the Tomita-Takesaki theorem (Connes&Rovelli).
- more: the off-shell generalization of algebraic observable function leads to a partition function description which closely resamble the path integral approach.

=> apply to general relativity!

# Gauge theory viewpoint: 4-d Palatini

the operational/relational content of Einstein's gravity is most evident in the passage from the metric to the first order Palatini formulation

$$S[e,\omega] = \int \epsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL} \qquad EE \begin{cases} d^D e^I = de^I + \omega^I{}_J \wedge e^J = 0 \\ \epsilon_{IJKL}\, e^J \wedge F^{KL} = 0 \end{cases}$$

- tetrad field $e \in \Omega^1(M, \mathbb{R}^{(1,3)})$, a (Minkowski) vector-valued 1-form taking value in the so-called internal space $e : TM \to \mathbb{R}^{(1,3)}$
- local spin connection $\omega \in \Omega^1(M, \mathfrak{so}(1,3))$, a $\mathfrak{so}(1,3)$-valued Lie algebra one-form: $\omega : TM \to \mathfrak{so}(1,3)$

- covariant hamiltonian approach: extended configuration space

$$(x, e, \omega) \in \mathcal{C} = M \times \tilde{\mathcal{C}}$$

seen as a bundle on spacetime $M$, with $\mathfrak{iso}(1,3) = \mathbb{R}^{(1,3)} \oplus \mathfrak{so}(1,3)$.

associated bundle structure

$$\begin{array}{c} \tilde{\mathcal{C}} = \mathfrak{iso}(1,3) \otimes T^*M \\ \Big\downarrow \pi \\ M \end{array}$$

Thank You