Search for gravitational waves from known pulsars using F and G statistics

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Method used so far

Bayesian estimation of pulsar parameters from gravitational wave data (Dupuis and Woan 2005).

Upper limits on amplitude of gravitational wave emission obtained for many known pulsars by LSC using S5 LIGO data.

Spin down limit beaten for the Crab pulsar.

Gravitational wave signal from a spinning neutron star

Signal depends on the set of parameters:

$$\theta = (h_0, \phi_0, \psi, \iota, \underbrace{\mathrm{f}, \delta, \alpha}_{\text{For targeted searches assumed known from radio observations}})$$

$$s(t) = A_1\, h_1(t) + A_2\, h_2(t) + A_3\, h_3(t) + A_4\, h_4(t)$$

$$h_1(t) = a(t) \cos\phi(t), \quad h_2(t) = b(t) \cos\phi(t)$$

$$h_3(t) = a(t) \sin\phi(t), \quad h_4(t) = b(t) \sin\phi(t)$$

Amplitudes

$$h_{0+} = \frac{1}{2} h_0 (1 + \cos^2 \iota),$$

$$h_{0\times} = h_0 \cos \iota,$$

$$A_1 = h_{0+} \cos 2\psi \cos \phi_0 - h_{0\times} \sin 2\psi \sin \phi_0,$$

$$A_2 = h_{0+} \sin 2\psi \cos \phi_0 + h_{0\times} \cos 2\psi \sin \phi_0,$$

$$A_3 = -h_{0+} \cos 2\psi \sin \phi_0 - h_{0\times} \sin 2\psi \cos \phi_0,$$

$$A_4 = -h_{0+} \sin 2\psi \sin \phi_0 + h_{0\times} \cos 2\psi \cos \phi_0.$$

Matched filter

Likelihood ratio test

$$\Lambda(x) := \frac{p_1(x)}{p_0(x)} \geq \lambda_0$$

Gaussian case

$$\ln \Lambda[x(t)] \cong 2\frac{T_\circ}{S_0}\left(\langle x(t)s(t)\rangle - \frac{1}{2}\langle s(t)^2\rangle\right)$$

H - statistic

$$\langle g \rangle := \frac{1}{T_\circ}\int_0^{T_\circ} g(t)\,\mathrm{d}t$$

$$\mathcal{H} := \langle x(t)s(t)\rangle \geq \mathcal{H}_0$$

Bayesian approach and composite hypothesis testing

$$\frac{\int_{\Theta} p_1(x;\boldsymbol{\theta})\pi(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{\theta}}{p_0(x)} \geq \gamma_o$$

For some pulsars like Crab and Vela parameters ψ, ι are estimated to a very good accuarcy from X-ray observations and only h_o and ϕ_o are unknown

Applying Bayesian approach to the case of the phase parameter ϕ_o :

$$\pi(\phi_o) = \frac{1}{2\pi}, \quad \phi_o \in [0, 2\pi)$$

For Gaussian case p_1 can be integrated over ϕ_o explicitly

A. D. Whalen, *Detection of Signals in Noise*, Academic Press (1971)

The G-statistic

$$\exp\Big(-\frac{h_0^2 N T_\mathrm{o}}{S_0}\Big)\, I_0\Big(2h_0\sqrt{\frac{T_\mathrm{o}N}{S_0}\mathcal{G}}\Big) \geq \gamma_0$$

G - statistic

$$\mathcal{G} \cong \frac{T_\mathrm{o}}{N S_0}\Big(\langle x(t)h_c(t)\rangle^2 + \langle x(t)h_s(t)\rangle^2\Big)$$

$$\mathcal{G} \geq \mathcal{G}_o$$

A uniformly most powerful test with respect to the amplitude h_o

G-statistic as a maximum likelihood test

Maximize the likelihood function with respect to h_o and ϕ_o

$$s(t) = h_o \cos\phi_o h_c(t) + h_o \sin\phi_o h_s(t)$$

Introduce the parameters A_c and A_s

$$A_c = h_o \cos\phi_o, \quad A_s = h_o \sin\phi_o$$

Maximum likelihood estimators of A_c and A_s

$$\hat{A}_c \cong \frac{\langle x h_c \rangle}{N} \qquad \hat{A}_s \cong \frac{\langle x h_s \rangle}{N}$$

$$\mathcal{G} \geq \mathcal{G}_o$$

A maximum likelihood test

The F-statistic

(A maximum likelihood test)

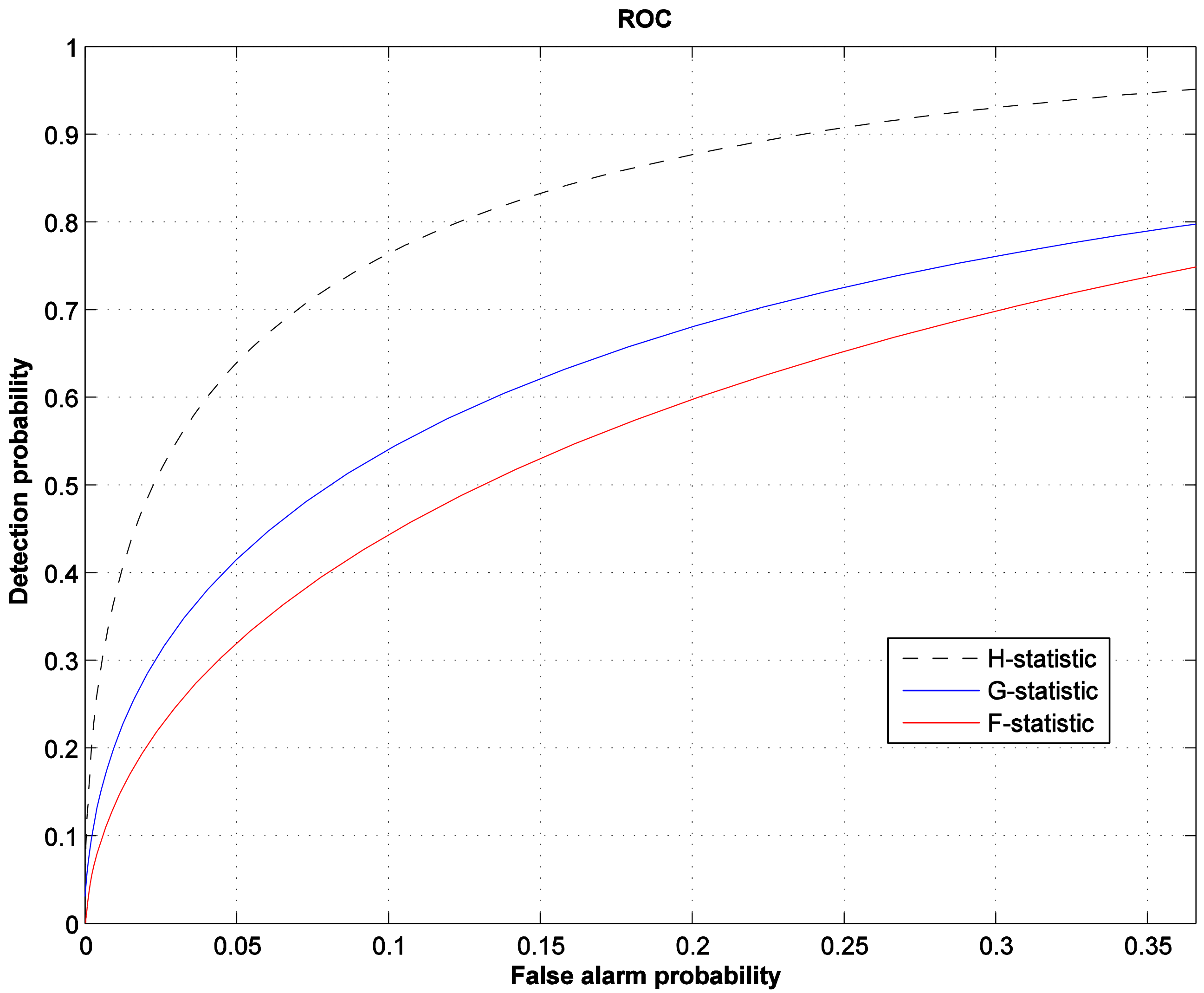
All the parameters h_o, ϕ_o, ψ, ι are unknown

$$s(t) = A_1 h_1(t) + A_2 h_2(t) + A_3 h_3(t) + A_4 h_4(t)$$

F - statistic

$$\mathcal{F} \cong \frac{2T_o}{S_0} \frac{B(\langle xh_1\rangle^2 + \langle xh_3\rangle^2) + A(\langle xh_2\rangle^2 + \langle xh_4\rangle^2) - 2C(\langle xh_1\rangle\langle xh_2\rangle + \langle xh_3\rangle\langle xh_4\rangle)}{D}$$

$$\mathcal{F} \geq \mathcal{F}_o$$



B-statistic (R. Prix) obtained using Bayesian approach can be better than the F-statistic for some reasonable a priori distributions of parameters.

Accuracy of ML parameter estimators

(Fisher matrix)

Only h_o and ϕ_o unknown (G-statistic search)

h_o and ϕ_o uncorrelated

$$\frac{\sigma_{h_o}}{h_o} = \frac{1}{\rho} \qquad \sigma_{\phi_o} = \frac{1}{\rho}$$

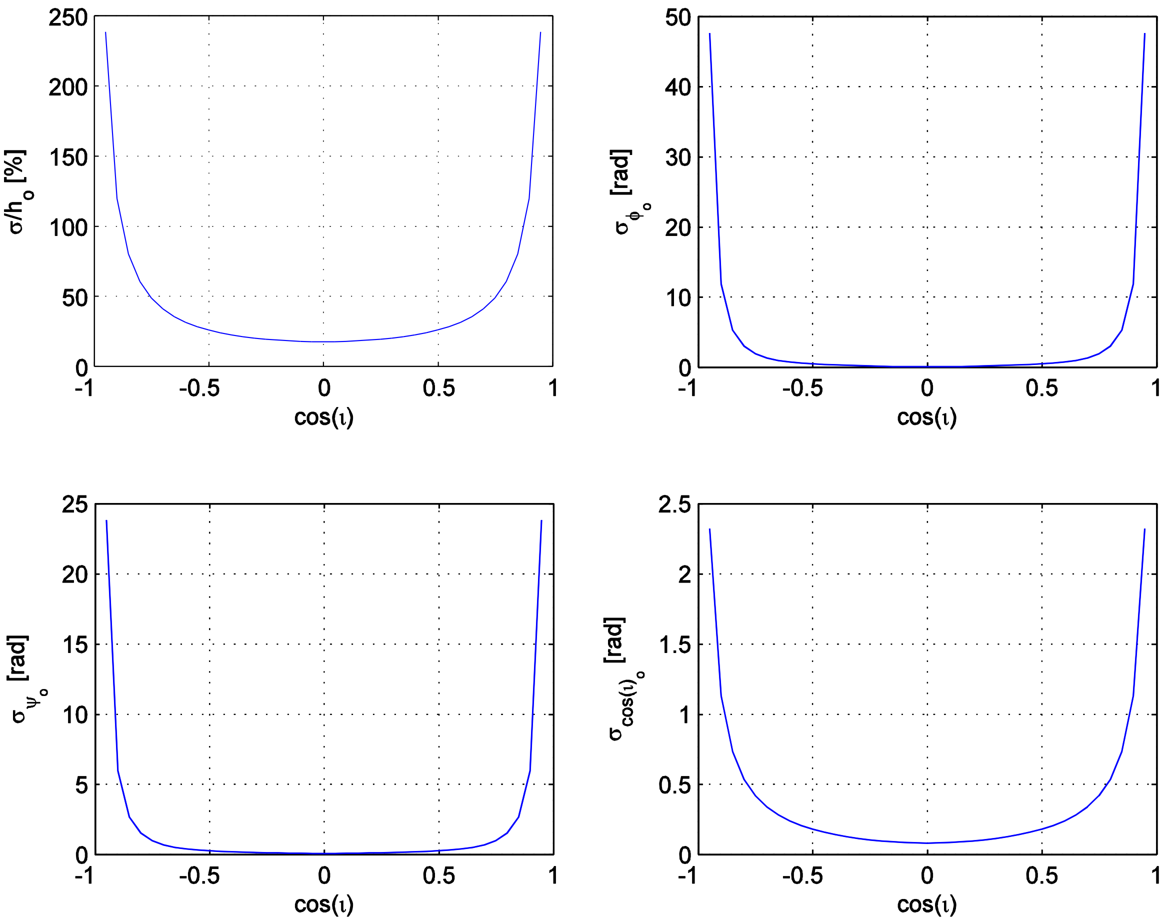
All h_o, ϕ_o, ψ, ι unknown (F-statistic search)

h_o, ϕ_o, ψ, ι correlated

$\sigma_{h_o}/h_o, \sigma_{\phi_o}, \sigma\ , \sigma_\iota$ are inversely proportional to signal-to-ratio ρ and depend on h_o only through ρ

Independent of ϕ_o
Very weakly dependent on ψ
Strongly dependent on ι

Dependence on ι



Singularity for a circularly polarized wave

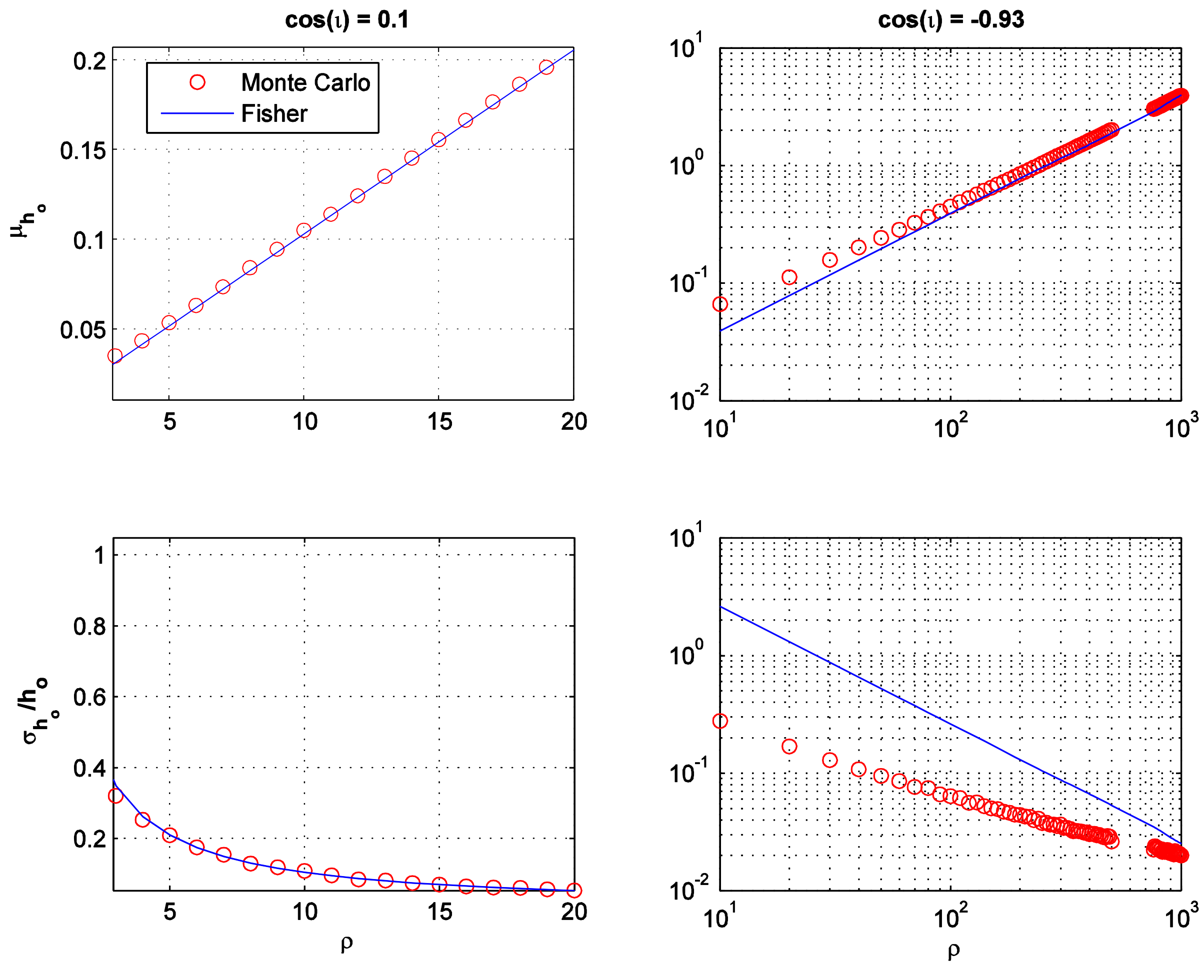
Singularity explained

$$\cos(\iota) = \pm 1$$

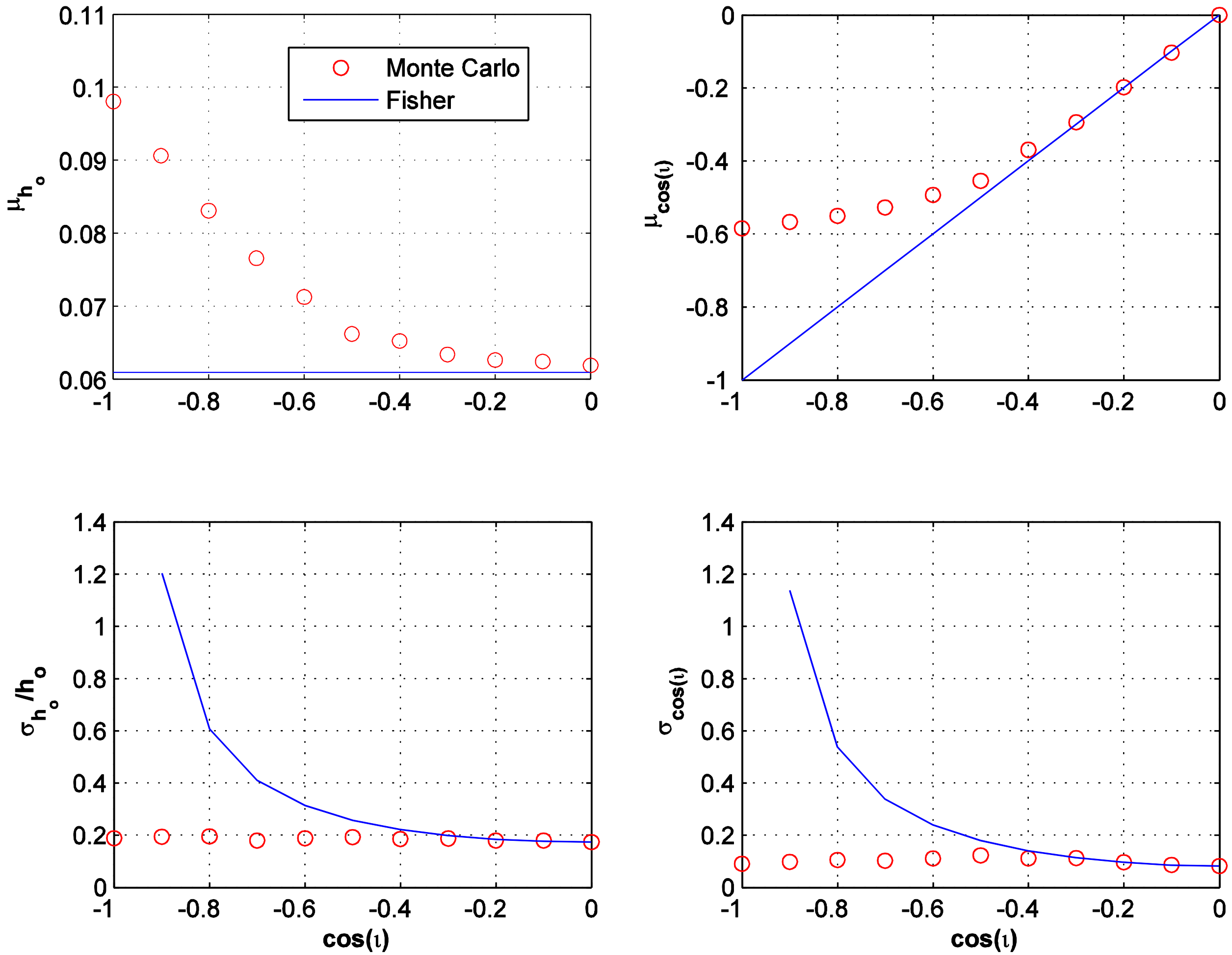
$$\begin{aligned}
A_{o1} &= h_o \cos(2\,\psi \pm \phi_o) \\
A_{o2} &= h_o \sin(2\,\psi \pm \phi_o) \\
A_{o3} &= \mp A_{o2} \\
A_{o4} &= \pm A_{o1}
\end{aligned}$$

Only 2 amplitudes independent. Determinant of the Fisher matrix is 0 and consequently a singularity for variances of all 4 parameters.

Fisher vs. Monte Carlo



Singularity regularized



X-ray observations vs GW

(Ng & Romani (2007))

Polarization (ψ) and inclination (ι) angles obtained from a torus model of pulsar wind nebula

SNR = 10	Ψ [rad]	σ_X [rad]	σ_GW [rad]	Cos(I)	σ_X	σ_GW
Crab	2.164	0.002	0.210	0.480	0.002	0.161
Vela	2.280	0.001	0.174	0.44	0.01	0.14