A method for detection of known sources of continuous gravitational wave signals in non-stationary data

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Outline of the talk

- Data preparation
- Formulation of the problem
- The signal and the noise
- The algebra of the 5-vectors
- The detection problem
- The parameter estimation problem
- The Coherence: reliability of the detection
- Extension to sub-periods or more antennas analysis

Data preparation

- Band extraction, by constructing the analytic signal, over bands of ~0.1Hz

- Doppler and spin-down correction

- Removal of residual time periods which are particularly noisy

- Wiener weighting: we weight the noise data with the inverse of their local variance, to reduce the effect of non-stationary noise

The formulation of the problem

There are two cases:

- we know the polarization parameter and so we should estimate the amplitude and the phase of the wave (2 d.o.f. problem). The detection of the phase and amplitude modulated signal is done with a matched filter in the space of the five signal Fourier components.

- we don't know the polarization parameters, so we should estimate also them (4 d.o.f. problem). The detection is more complex.

The wave

$$\kappa_+^2+\kappa_\times^2=1,\text{real}$$

$$\mathrm{h}(t)=h_0\cdot\left(\mathbf{e}_\oplus\cdot\kappa_+ +\mathbf{e}_\otimes\cdot\kappa_\times\exp\left(j\varphi\right)\right)\cdot e^{\left(j\cdot\left(\omega_0 t+\gamma\right)\right)}$$

tensors in arial

If the polarization ellipse has semi-axes a ≥ b, with the convention to put b positive if the circular part is L (CCW) and negative if it is R (CW).

The two polarization parameters are $\eta=\dfrac{b}{a}$, defined in the range -1≤η≤1 (0 if linear polarization)

and the polarization angle ψ (direction of the axis a of the ellipse)

$$\boxed{\begin{gathered}\mathrm{h}(t)=h_0\cdot\left(H_+\cdot\mathbf{e}_\oplus+H_\times\cdot\mathbf{e}_\otimes\right)\cdot e^{\left(j\cdot\left(\omega_0 t+\gamma\right)\right)}\\ H_+=\frac{1}{\sqrt{1+\eta^2}}\cdot\left(\cos\left(2\psi\right)-j\cdot\eta\cdot\sin\left(2\psi\right)\right)\\ H_\times=\frac{1}{\sqrt{1+\eta^2}}\cdot\left(\sin\left(2\psi\right)+j\cdot\eta\cdot\cos\left(2\psi\right)\right)\end{gathered}}$$

Invariants and inverse problem

$$H_+^* = H_+ \cdot e^{j\gamma} \qquad H_\times^* = H_\times \cdot e^{j\gamma}$$

Invariants: functions which do not depend on the unknown parameter γ

First invariant

$$H_+^* \cdot H_\times^{*\prime} = H_+ \cdot H'_\times =$$

$$= \frac{1-\eta^2}{1+\eta^2} \cdot \frac{1}{2} \cdot \sin(4\psi) + j \cdot \frac{-\eta}{1+\eta^2} = A + jB$$

Second invariant

$$\left|H_+^*\right|^2 - \left|H_\times^*\right|^2 = \left|H_+\right|^2 - \left|H_\times\right|^2 =$$

$$= \frac{1-\eta^2}{1+\eta^2} \cdot \cos(4\psi) = C$$

Inversion

$$\boxed{\eta = \frac{-1+\sqrt{1-4B^2}}{2B} \qquad \cos(4\psi) = \frac{C}{\sqrt{(2A)^2 + C^2}} \qquad \sin(4\psi) = \frac{2A}{\sqrt{(2A)^2 + C^2}}}$$

The antenna response

fast modulation

$$h(t) = h_0 \cdot \left(A_+ \cdot H_+ + A_\times \cdot H_\times\right) \cdot \exp\left(j\left(\omega_0 t + \gamma\right)\right)$$

slow modulation (phase and amplitude)

$$A_+ = a_0 + a_{1c} \cdot \cos\left(\Omega \cdot t\right) + a_{1s} \cdot \sin\left(\Omega \cdot t\right) + a_{2c} \cdot \cos\left(2 \cdot \Omega \cdot t\right) + a_{2s} \cdot \sin\left(2 \cdot \Omega \cdot t\right)$$

$$A_\times = b_{1c} \cdot \cos\left(\Omega \cdot t\right) + b_{1s} \cdot \sin\left(\Omega \cdot t\right) + b_{2c} \cdot \cos\left(2 \cdot \Omega \cdot t\right) + b_{2s} \cdot \sin\left(2 \cdot \Omega \cdot t\right)$$

$$a_0 = -\frac{3}{16}\left(1 + \cos 2\delta\right)\left(1 + \cos 2\lambda\right)\cos 2a$$

$$a_{1c} = -\frac{1}{4}\sin 2\delta \sin 2\lambda \cos 2a$$

$$a_{1s} = -\frac{1}{2}\sin 2\delta \cos\lambda \sin 2a$$

$$a_{2c} = -\frac{1}{16}\left(3 - \cos 2\delta\right)\left(3 - \cos 2\lambda\right)\cos 2a$$

$$a_{2s} = -\frac{1}{4}\left(3 - \cos 2\delta\right)\sin\lambda \sin 2a$$

$$b_{1c} = -\cos\delta \cos\lambda \sin 2a$$

$$b_{1s} = \frac{1}{2}\cos\delta \sin 2\lambda \cos 2a$$

$$b_{2c} = -\sin\delta \sin\lambda \sin 2a$$

$$b_{2s} = \frac{1}{4}\sin\delta\left(3 - \cos 2\lambda\right)\cos 2a$$

(α,δ) source coordinates, λ and a the latitude and azimuth of the antenna;
$\Omega \cdot t = \alpha - \Theta$, where Θ is the sidereal time.

The 5-vectors (the signal)

The "generator" 5-vector:
from 5 complex numbers it generates the time response

$$W_k = e^{jk\Omega t} \quad \text{with} \quad -2 \le k \le 2$$

The two polarization signals 5-vectors **A⁺** and **Aˣ** with components:

$$A^{+}_{-2} = \frac{a_{2c}}{2} - j\frac{a_{2s}}{2} \qquad A^{\times}_{-2} = \frac{b_{2c}}{2} - j\frac{b_{2s}}{2}$$

$$A^{+}_{-1} = \frac{a_{1c}}{2} - j\frac{a_{1s}}{2} \qquad A^{\times}_{-1} = \frac{b_{1c}}{2} - j\frac{b_{1s}}{2}$$

$$A^{+}_{0} = a_0 \qquad A^{\times}_{0} = 0$$

$$A^{+}_{1} = \frac{a_{1c}}{2} + j\frac{a_{1s}}{2} \qquad A^{\times}_{1} = \frac{b_{1c}}{2} + j\frac{b_{1s}}{2}$$

$$A^{+}_{2} = \frac{a_{2c}}{2} + j\frac{a_{2s}}{2} \qquad A^{\times}_{2} = \frac{b_{2c}}{2} + j\frac{b_{2s}}{2}$$

The signal 5-vector

$$\boldsymbol{A} = H_{+}\boldsymbol{A}^{+} + H_{\times}\boldsymbol{A}^{\times}$$

The antenna response (analytic signal)

$$h(t) = h_0 \cdot \boldsymbol{A} \cdot \boldsymbol{W} \cdot \exp\left(j \cdot (\omega_0 t + \gamma)\right)$$

scalar product

Basic problems:

Detect the presence of the signal

Signal parameters estimation

Reliability of the detection

Extension of the procedure to more antennas and/or to the analysis in sub-periods of time

The 5-vectors (the data)

$$x(t) = h(t) + n(t)$$

The data 5-vector

$$\boldsymbol{X} = \int_T x(t) \cdot \boldsymbol{W'} \cdot \exp(-j\omega_0 t) \cdot dt =$$

$$= \int_T x(t) \cdot \exp\left(-j(\omega_0 - \boldsymbol{k} \cdot \Omega) \cdot t\right) \cdot dt =$$

$$= h_0 e^{j\gamma} \boldsymbol{A} + \boldsymbol{N}$$

The "real signal" 5-vector

$$\boldsymbol{A} = \int_{obs.per.} w(t) \cdot s(t) \cdot \exp\left(-j(\omega_0 - \boldsymbol{k} \cdot \Omega) \cdot t\right) dt$$

w(t) is the "Wiener filter", if it is used in order to optimize for the non-stationarity of the noise.

The 5-vectors (the signal)

$$\boldsymbol{A} = \int_{obs.per.} w(t) \cdot s(t) \cdot \exp\left(-j\left(\omega_0 - \boldsymbol{k} \cdot \Omega\right) \cdot t\right) dt$$

The 5-vector of the signal could in principle be constructed analytically from this equation , but this is not what we do, because it is important to generate a signal with the whole procedure applied to the data, that is the same cuts, vetoes, Wiener weights.

We thus create the 5-vect $\boldsymbol{S}$ (case of 2 d.o.f.) or its two plus and cross components (case of 4 d.o.f.) and then operate on it exactly as we do on the noise.

Why use the 5-vectors ?

❖It is immediate to understand that operating with this framework gives a big gain in computation speed. This is mainly in the case of the simulation of many source injections, not precise frequency search (in fact the signal 5-vect does not depend on the frequency), different analysis, and so on.

❖In the case of the matched filter for the Vela pulsar during VSR1 (the reduced data were about 10^7 samples) the gain is of the order of 1 million. And without any loss in performances.

❖We should compute at the beginning only the three 5-vectors: the two for the basic signals (plus and cross) and one for the data.

Data 5-vect: the Fourier components at the 5 frequencies
Noise background 5-vect: from one FFT and a comb with the 5 lines we get many noise realization
Signal 5-vect: a linear combination of the two plus and cross 5 vectors:

$$\boldsymbol{A} = H_{+}\boldsymbol{A}^{+} + H_{\times}\boldsymbol{A}^{\times}$$

The detection (2 d.o.f. case)

The matched filter is

$$h = \frac{\boldsymbol{X} \cdot \boldsymbol{A}}{|\boldsymbol{A}|^2}$$

where **X** is the 5-vect of the data and **A** is the 5-vect of the signal

Note that the filter $\dfrac{\boldsymbol{A}}{|\boldsymbol{A}|^2}$ is a 5-vector.

The signal at the filter output is only one complex number.

The detection (4 d.o.f. case)

$$\mathrm{h}(t) = h_0 \cdot \left(\mathrm{e}_{\oplus} \cdot a + \mathrm{e}_{\otimes} \cdot b\right) \cdot \exp\left(j\omega_0 t\right) \qquad \sqrt{|a|^2 + |b|^2} = 1$$

The two basic observables are $h_+ = \dfrac{\boldsymbol{X} \cdot \boldsymbol{A}^+}{\left|\boldsymbol{A}^+\right|^2}$ $\quad h_\times = \dfrac{\boldsymbol{X} \cdot \boldsymbol{A}^\times}{\left|\boldsymbol{A}^\times\right|^2}$

$$h = \frac{\boldsymbol{X} \cdot \boldsymbol{A}}{|\boldsymbol{A}|^2} = \frac{\boldsymbol{X} \cdot \left(a\boldsymbol{A}^+ + b\boldsymbol{A}^\times\right)}{|a|^2 \left|\boldsymbol{A}^+\right|^2 + |b|^2 \left|\boldsymbol{A}^\times\right|^2} =$$

$$= \frac{\left|\boldsymbol{A}^+\right|^2 \cdot a}{|a|^2 \left|\boldsymbol{A}^+\right|^2 + |b|^2 \left|\boldsymbol{A}^\times\right|^2} \cdot h_+ + \frac{\left|\boldsymbol{A}^\times\right|^2 \cdot b}{|a|^2 \left|\boldsymbol{A}^+\right|^2 + |b|^2 \left|\boldsymbol{A}^\times\right|^2} \cdot h_\times$$

The matched filter h is the weighted mean of the two basic components

But we don not know a and b !

Detection statistics: basic observables

Basic observables

$$h_+ = \frac{\boldsymbol{X} \cdot \boldsymbol{A}^+}{\left|\boldsymbol{A}^+\right|^2} \qquad h_\times = \frac{\boldsymbol{X} \cdot \boldsymbol{A}^\times}{\left|\boldsymbol{A}^\times\right|^2}$$

Variances in case of only noise

$$\sigma_+^2 = \frac{\sigma_X^2}{\left|\boldsymbol{A}^+\right|^2} \qquad \sigma_\times^2 = \frac{\sigma_X^2}{\left|\boldsymbol{A}^\times\right|^2}$$

The distribution of a single observable (only noise)

$$f(x) = \frac{\left|\boldsymbol{A}^{+/\times}\right|^2}{\sigma_X^2} e^{-\frac{\left|\boldsymbol{A}^{+/\times}\right|^2}{\sigma_X^2} \cdot x}$$

The distribution of a single observable (noise+signal λ)

$$f(x;2,\lambda) = \frac{1}{2} e^{-(x+\lambda)/2} \cdot I_0\left(\sqrt{\lambda x}\right)$$

Detection statistics and its optimization

The detection statistics is a linear combination of the 2 observables

$$S = c_+ \cdot |h_+|^2 + c_\times \cdot |h_\times|^2$$

Some cases

simple mean **(blue)** $c_+ = c_\times = \frac{1}{2}$

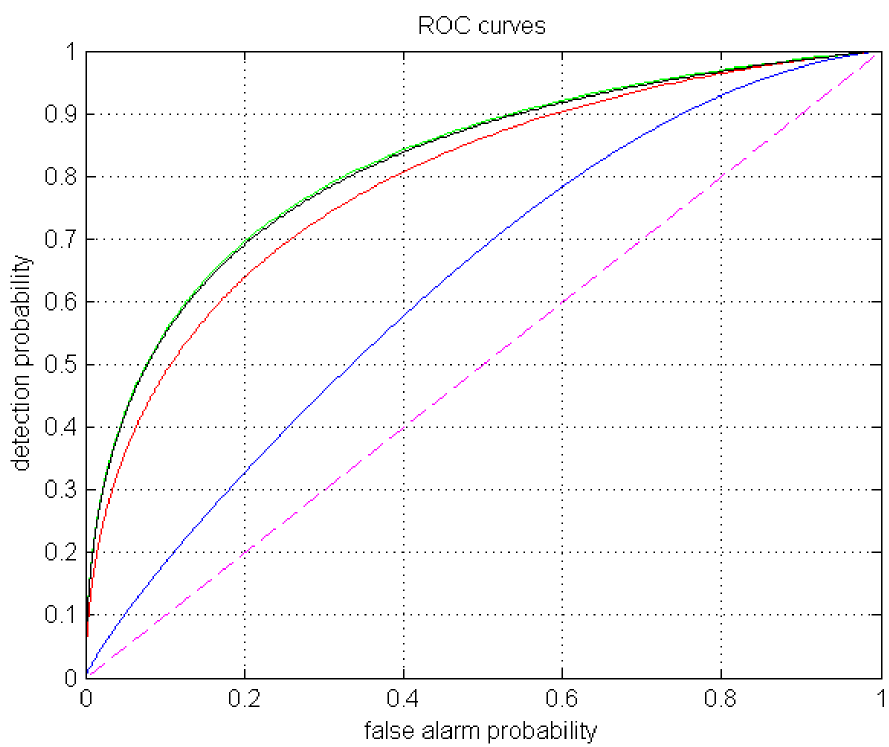
F statistics **(red)** $c_+ = |\boldsymbol{A}^+|^2$ $c_\times = |\boldsymbol{A}^\times|^2$

Best ROC statistics **(green)** $c_+ = |\boldsymbol{A}^+|^4$ $c_\times = |\boldsymbol{A}^\times|^4$

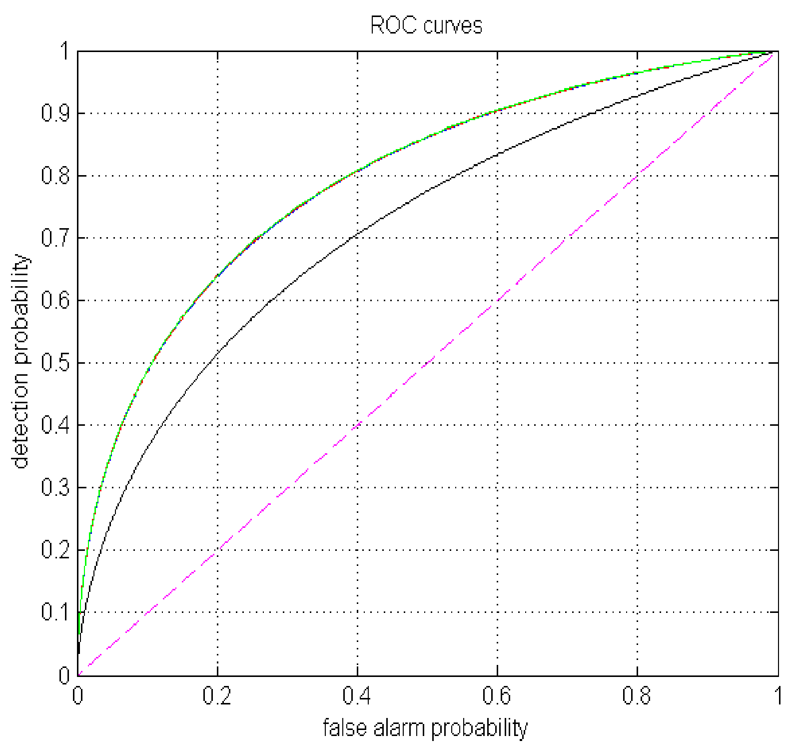
Best SNR statistics **(black)** $c_{+/\times} = 1$ $c_{\times/+} = 0$

Detection statistics

simple mean **(blue)** F statistics **(red)**
Best ROC statistics **(green)** Best SNR statistics **(black)**



Ratio of the two modes: 3



Ratio of the two modes: 1

Detection statistics: the distribution

Exponential distributions mixtures:

$$f(x) = \frac{\alpha\beta}{\alpha - \beta} \cdot \left(e^{-\beta x} - e^{-\alpha x}\right)$$

if $\alpha=\beta$

$$f(x) = \alpha^2 \cdot x \cdot e^{-\alpha \cdot x}$$

$\alpha=\beta$ is the F-stat case, that is equalization of the two plus and cross modes

Estimation of source parameters: basic relations

$$\boldsymbol{X} = h_0 e^{j\gamma} \cdot \left(a\boldsymbol{A}^{+} + b\boldsymbol{A}^{\times}\right) + \boldsymbol{N}$$

$$E\left[h_{+}\right] = E\left[\frac{\boldsymbol{X}\boldsymbol{A}^{+\prime}}{\left|\boldsymbol{A}^{+}\right|^{2}}\right] = E\left[\frac{\left[h_0 \cdot \left(a\boldsymbol{A}^{+} + b\boldsymbol{A}^{\times}\right) + \boldsymbol{N}\right] \cdot \boldsymbol{A}^{+\prime}}{\left|\boldsymbol{A}^{+}\right|^{2}}\right] = h_0 e^{j\gamma} \cdot a$$

$$E\left[h_{\times}\right] = E\left[\frac{\boldsymbol{X}\boldsymbol{A}^{\times\prime}}{\left|\boldsymbol{A}^{\times}\right|^{2}}\right] = E\left[\frac{\left[h_0 \cdot \left(a\boldsymbol{A}^{+} + b\boldsymbol{A}^{\times}\right) + \boldsymbol{N}\right] \cdot \boldsymbol{A}^{\times\prime}}{\left|\boldsymbol{A}^{\times}\right|^{2}}\right] = h_0 e^{j\gamma} \cdot b$$

$$a = \frac{h_{+} \cdot e^{-j\gamma}}{\sqrt{\left|h_{+}\right|^{2} + \left|h_{\times}\right|^{2}}} \qquad b = \frac{h_{\times} \cdot e^{-j\gamma}}{\sqrt{\left|h_{+}\right|^{2} + \left|h_{\times}\right|^{2}}}$$

$$h_0 = \frac{\left(\left|\boldsymbol{A}^{+}\right|^{2} \cdot \left|h_{+}\right|^{2} + \left|\boldsymbol{A}^{\times}\right|^{2} \cdot \left|h_{\times}\right|^{2}\right)\left(\left|h_{+}\right|^{2} + \left|h_{\times}\right|^{2}\right)}{\sqrt{\left|h_{+}\right|^{2} + \left|h_{\times}\right|^{2}} \cdot \left(\left|h_{+}\right|^{2}\left|\boldsymbol{A}^{+}\right|^{2} + \left|h_{\times}\right|^{2}\left|\boldsymbol{A}^{\times}\right|^{2}\right)} = \sqrt{\left|h_{+}\right|^{2} + \left|h_{\times}\right|^{2}}$$

Estimation of source parameters

Estimation of the amplitude $$\hat{h}_0 = \sqrt{|h_+|^2 + |h_\times|^2}$$

Invariants $$\hat{h}_+ \cdot \hat{h}_\times ' = A + jB \qquad |\hat{h}_+|^2 - |\hat{h}_\times|^2 = C$$

Estimation of η $$\eta = \frac{-1 + \sqrt{1 - 4B^2}}{2B}$$

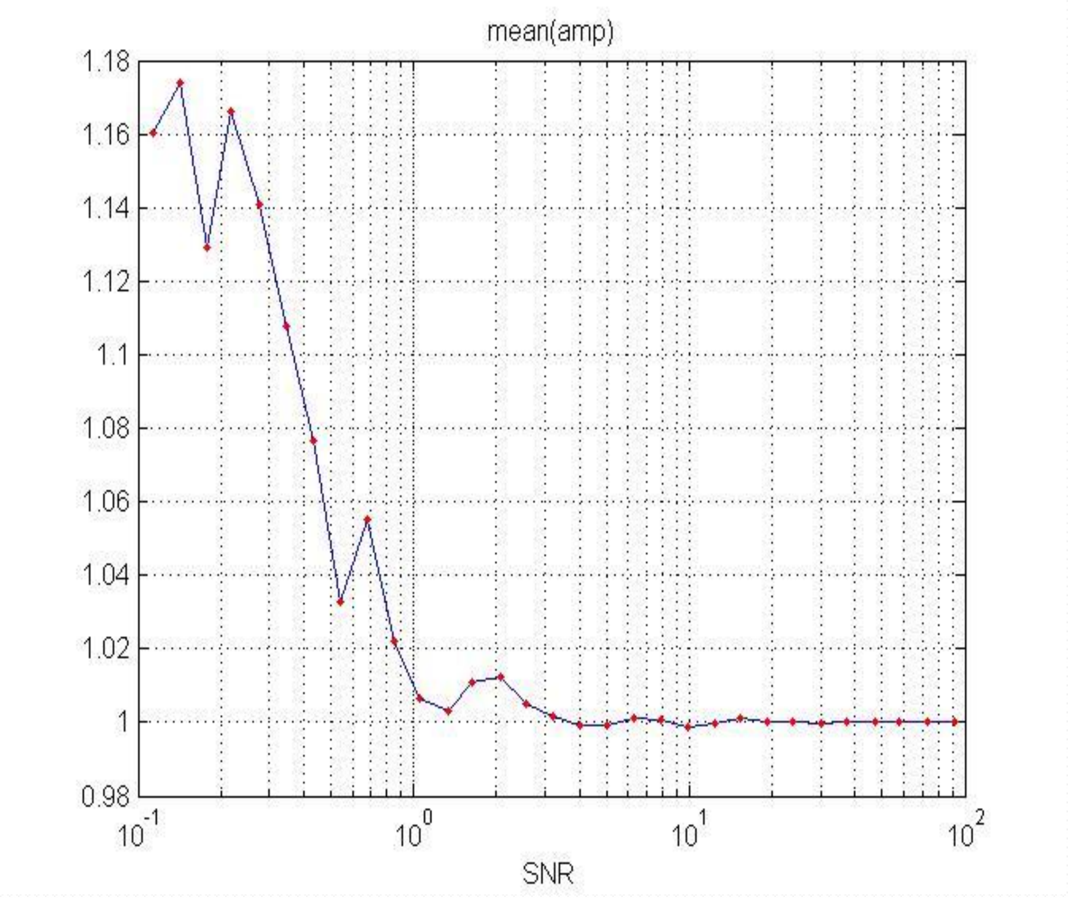
Estimation of ψ
$$\cos(4\psi) = \frac{C}{\sqrt{(2A)^2 + B^2}}$$
$$\sin(4\psi) = \frac{2A}{\sqrt{(2A)^2 + B^2}}$$

(these are independent on the absolute phase γ)

Estimation of the absolute phase $$e^{j\gamma} = \frac{\hat{h}_{+/\times}^{(sperim.)}}{\hat{h}_{+/\times}^{(teor.\ \gamma=0)}}$$

Estimation of the source parameters

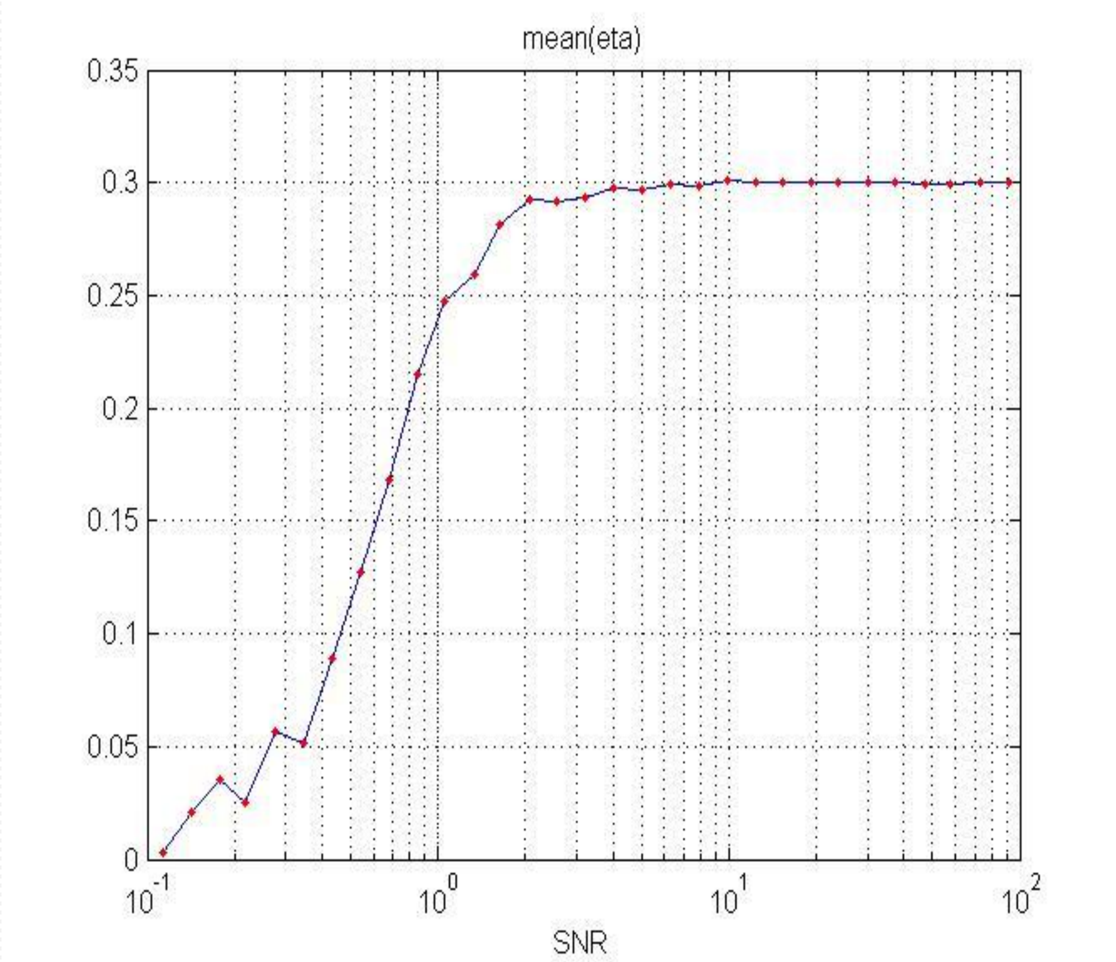
Simulation for $(\eta,\psi)=(0.3,30^0)$ Antenna latitude 5^0 Source declination -5^0



Amplitude

Estimation of the source parameters

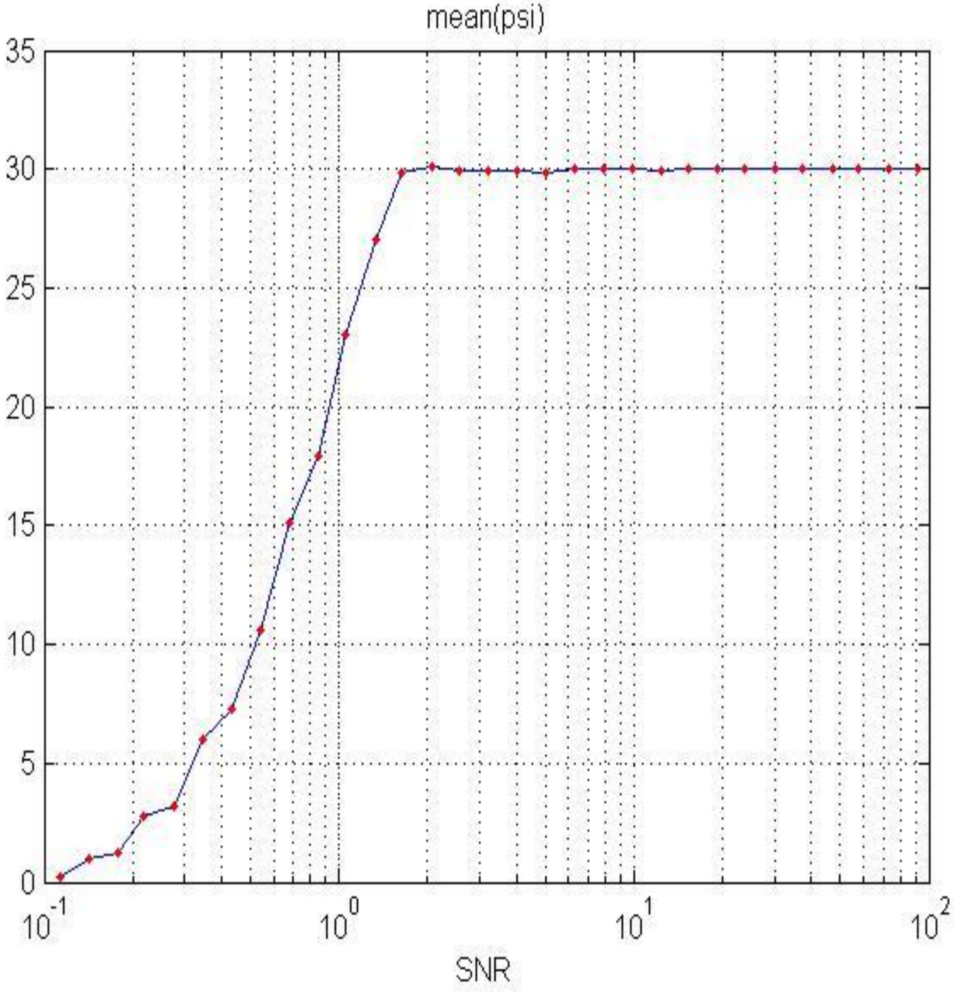
Simulation for $(\eta,\psi)=(0.3,30^0)$ Antenna latitude 5^0 Source declination -5^0



$\eta=0.3$

Estimation of the source parameters

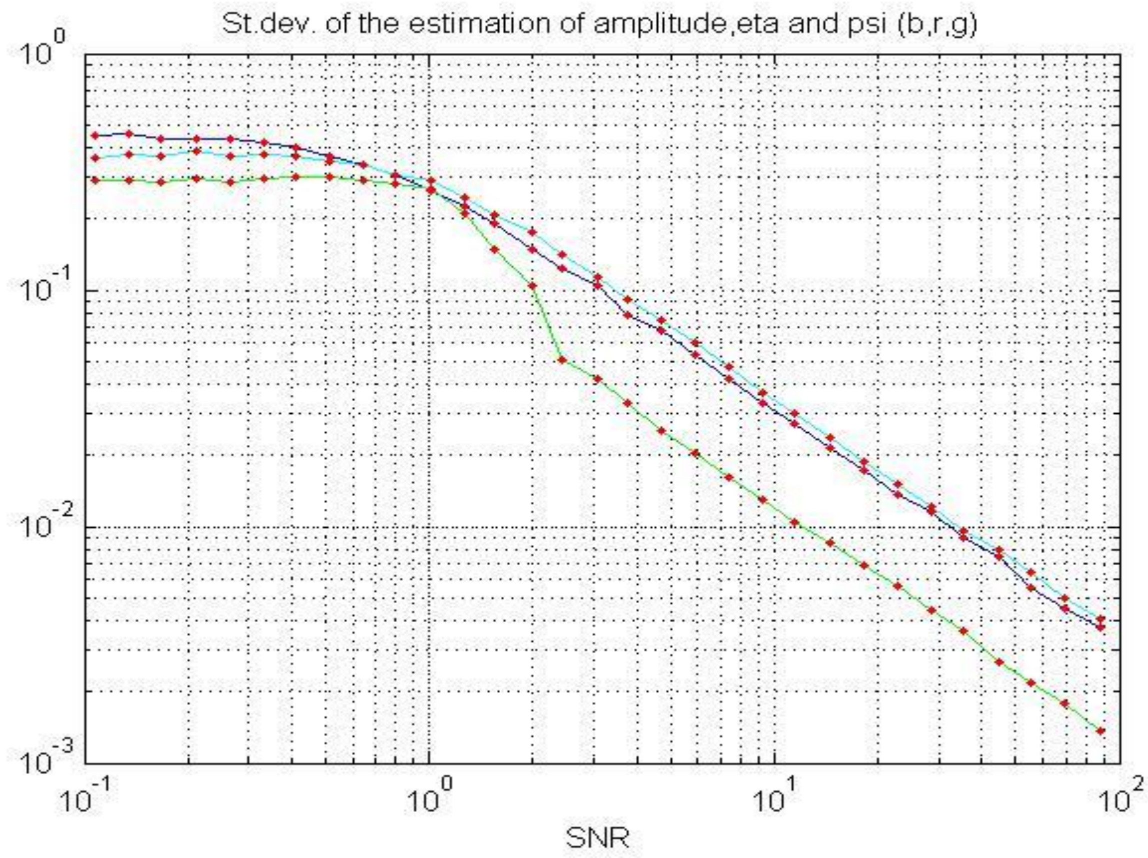
Simulation for $(\eta,\psi)=(0.3,30^0)$ Antenna latitude 5^0 Source declination -5^0



ψ=30 deg

Estimation of the source parameters

Simulation for $(\eta,\psi)=(0.3,30^0)$ Antenna latitude 5^0 Source declination -5^0



Standard deviations

ψ green
η red
amp blue

Reliability of the detection: the coherence

$$c = \frac{\left| g\boldsymbol{S} \right|^2}{\left| \boldsymbol{X} \right|^2}$$

Not the standard definition of coherence.
It indicates a coherence between the signal shape and the data.
It is not a function, but only a single number.

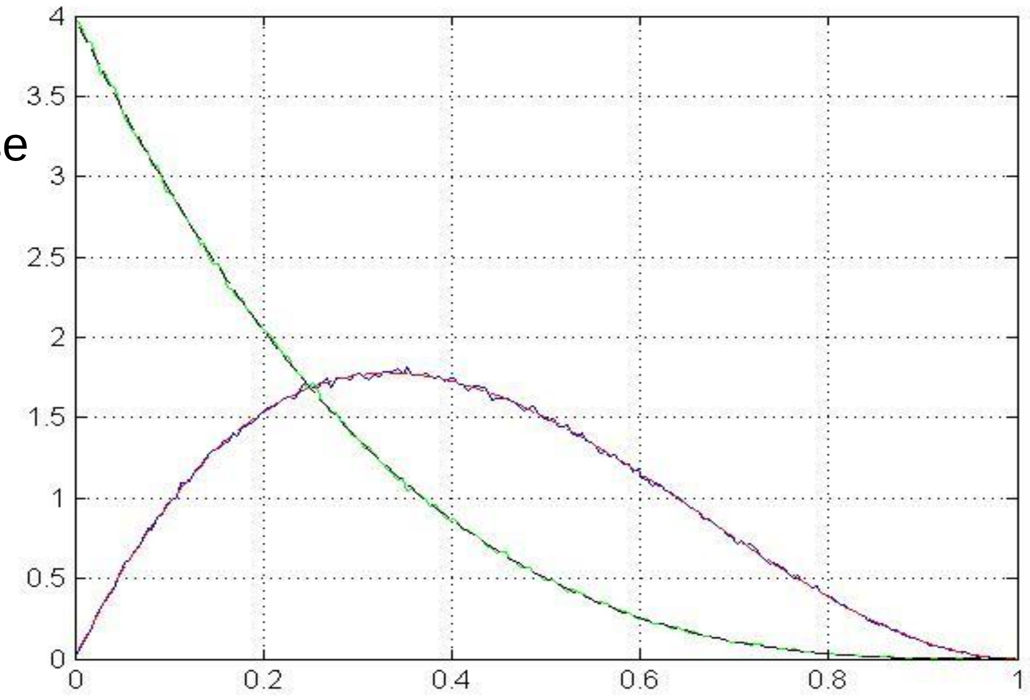
$$0 \leq c \leq 1$$

Noise only: it follows a beta distribution
(experimental result)

c does not depend on scaling factors on the signal. It depends only on the signal shape

Reliability of the detection: the coherence

Distribution
absence of noise



coherence

Green: 2 d.o.f.
Red: 4 d.o.f.

In the 2 d.o.f. case the coherence is more stringent, in the 4 d.o.f. case it is not so stringent

Extension of the method: use of the 5n-vectors

• Detection with more than one antenna: each antenna produces 5 components for the signal vector and for the data vector

• Sub-interval analysis: to enhance the reliability of the detection, the observation period can be divided in sub-periods, and each of them gives 5 components for the signal vector and for the data vector.

•It should be useful to divide the observation time into small pieces to get more information from the analysis: e.g. is the signal always present or not ?

Use of the 5n-vectors (both 2 and 4 d.o.f.)

$$\boldsymbol{X}^{(n)} = g \cdot \boldsymbol{S}^{(n)} + \boldsymbol{N}^{(n)}$$

are 5n vectors (data, signal and noise) and defining

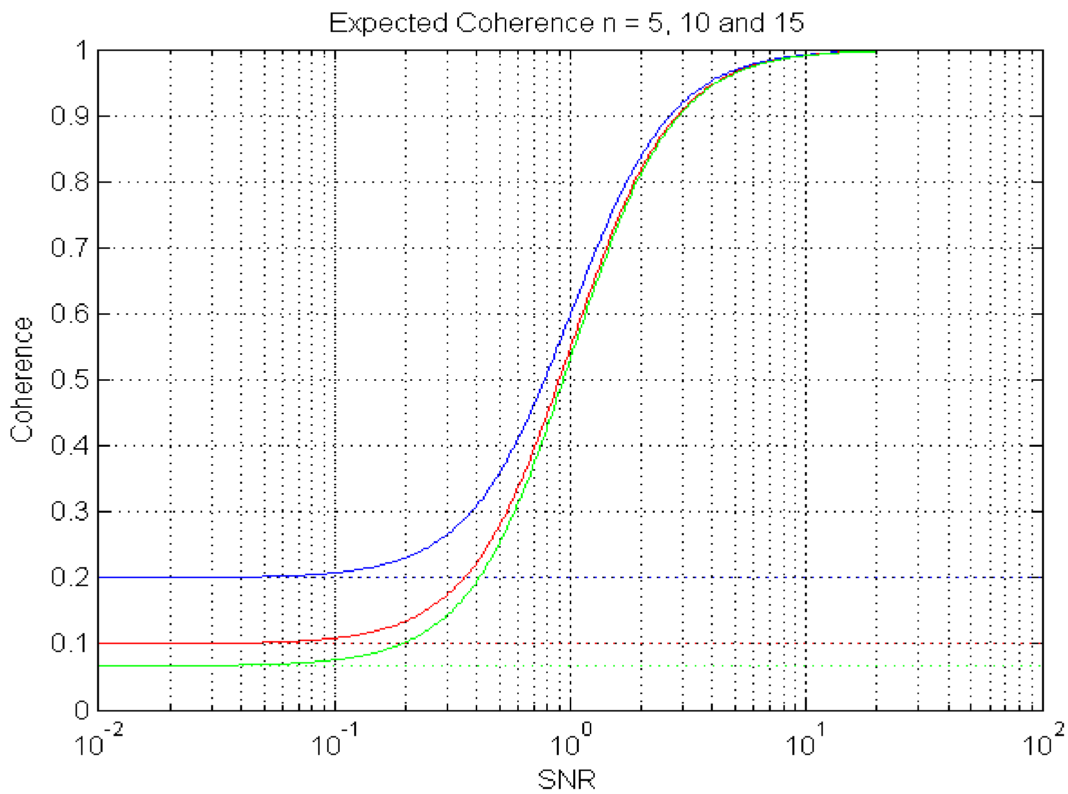
$$\rho^{(n)} = \frac{SNR}{|g|^2}$$

we have that

$$\rho^{(n)} = n \cdot \frac{\left|\boldsymbol{S}^{(n)}\right|^2}{\left|\boldsymbol{N}^{(n)}\right|^2} = \frac{\left|\boldsymbol{S}^{(n)}\right|^2}{\sigma^2_{N(n)}} = \frac{\left|\boldsymbol{S}^{(1)}\right|^2}{\sigma^2_{N(1)}} = \rho^{(1)} = \rho$$

ρ remains the same, while -see next slide-

the coherence, using 5n-vect, in absence of signal decreases with n



Blue: n=5
Red: n=10
Green: n=15

For SNR<<1 c →1/n; For high SNR c=1