"Super-sky" Parameter-Space Metric for Coherent Continuous-Wave Searches

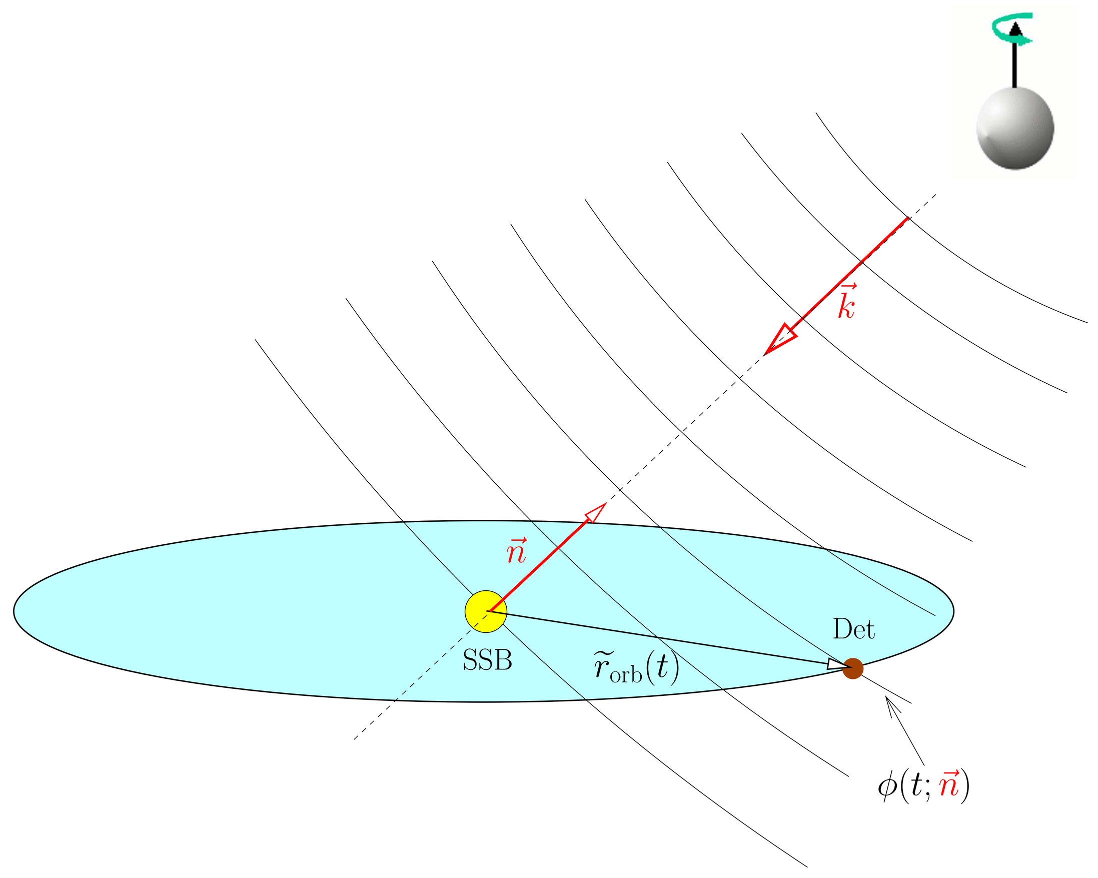
Reinhard Prix
Albert-Einstein-Institut Hannover, Germany

Abstract

The parameter-space metric of coherent searches for continuous gravitational waves is generally non-constant, due to the components involving the signal's sky position. This long-standing problem greatly complicates (i) template placement, (ii) coincidence schemes, and (iii) estimating the number of templates. We show that extending the 2D sky to a 3-dimensional "supersky" yields a constant metric in the embedding space. This formalism can help with (i)-(iii).

Definition of the CW metric

Unknown "Doppler" parameters λ: sky-location $\tilde{\mathbf{n}}$ of the source, intrinsic frequency f and spin-down parameters $\dot{f}$, ...
Note: $|\tilde{\mathbf{n}}| = 1$ is a unit-vector pointing to the sky-location of the source.



Signal phase $\Phi(t)$ is approximately [1]:

$$\Phi(t;\lambda) \approx 2\pi \left[f t + f \frac{\tilde{\mathbf{n}} \cdot \mathbf{r}(t)}{c} + \frac{1}{2} \dot{f} t^2 + \ldots \right] \tag{1}$$

The metric [2, 3] measures relative loss m of SNR^2 due to an offset $\Delta\lambda$ in Doppler-parameters from signal-location λ_s ("mismatch"):

$$m \equiv \frac{\mathrm{SNR}^2(\lambda_s) - \mathrm{SNR}^2(\lambda_s + \Delta\lambda)}{\mathrm{SNR}^2(\lambda_s)}$$

Taylor-expanding this for small offsets $\Delta\lambda$ defines the metric tensor g_{ij} :

$$m = g_{ij}(\lambda_s)\, \Delta\lambda^i \Delta\lambda^j + \mathcal{O}(\Delta\lambda^3)$$

The metric tensor g_{ij} can be computed approximately [4] as

$$g_{ij}(\lambda_s) \approx \langle \partial_i \Phi \, \partial_j \Phi \rangle - \langle \partial_i \Phi \rangle \langle \partial_j \Phi \rangle$$

in terms of phase-derivatives $\partial_i \Phi \equiv \partial\Phi / \partial\lambda^i$ and where $\langle Q \rangle$ is the time-average of $Q(t)$ over the observation time T:

$$\langle Q \rangle \equiv \frac{1}{T} \int_0^T Q(t)\, dt$$

Phase variation in natural units

We use rescaled quantities, namely

$$t \to \frac{t}{T} \qquad \mathbf{r}(t) \to \frac{\mathbf{r}(t)}{R_{\rm orb}}$$

measuring t in units of the observation time T and $\mathbf{r}$ in units of the orbital radius $R_{\rm orb}$. Similarly we rescale the Doppler parameters as

$$f^{(s)} \to \frac{2\pi}{(s+1)!} f^{(s)} T^{s+1}, \qquad \tilde{\mathbf{n}} \to \tilde{\mathbf{n}}' \equiv \frac{2\pi}{c} \bar{f}\, R_{\rm orb}\, \tilde{\mathbf{n}},$$

and so we can now write the variation $d\Phi(t;\lambda)$ of the signal-phase (1) as

$$d\Phi(t;\lambda) \approx t\, df + \mathbf{r}(t) \cdot d\tilde{\mathbf{n}}' + t^2\, d\dot{f} + \ldots \tag{2}$$

Metric structure

The metric g_{ij} has a block-structure in the 2 "sky"-components $\tilde{\mathbf{n}}$ and the $s_{\rm max}$ "spin" components $f^{(s)} \in \{f, \dot{f}, \ldots\}$:

$$g_{ij} = \begin{pmatrix} g_{ab} & g_{as'} \\ g_{sb} & g_{ss'} \end{pmatrix} = \begin{pmatrix} \boxed{\begin{matrix}\text{sky} \times \text{sky} \\ (2 \times 2)\end{matrix}} & \boxed{\begin{matrix}\text{sky} \times \text{spin} \\ (2 \times s_{\rm max})\end{matrix}} \\ \boxed{\begin{matrix}\text{spin} \times \text{sky} \\ (s_{\rm max} \times 2)\end{matrix}} & \boxed{\begin{matrix}\text{spin} \times \text{spin} \\ (s_{\rm max} \times s_{\rm max})\end{matrix}} \end{pmatrix}$$

Spin metric

The metric in the "spin" components $\lambda^s \in \{f, \dot{f}, \ldots\}$ is easy to compute analytically, and is found to be constant:

$$g_{ss'} = \langle t^s t^{s'} \rangle - \langle t^s \rangle \langle t^{s'} \rangle = \frac{(s+1)(s'+1)}{(s+2)(s'+2)(s+s'+3)}$$

Sky metric

The metric involving "sky" components $\tilde{\mathbf{n}}$ involves integrations over the detector-motion $\mathbf{r}(t)$, and generally varies over the sky:

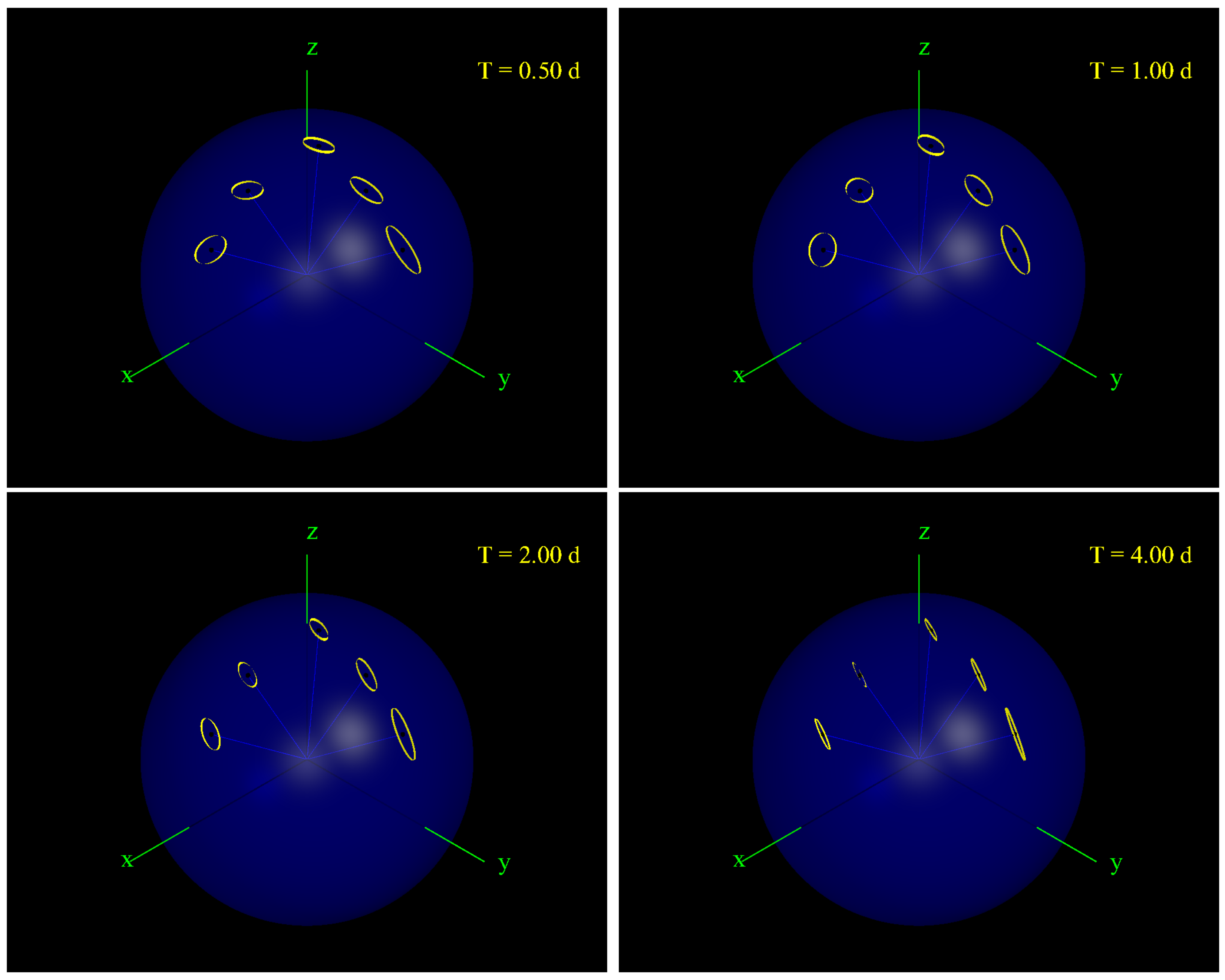


Figure: Sky iso-mismatch ellipses (yellow) over the sky-sphere (blue) $|\mathbf{n}| = 1$, for different observation times T, at fixed frequency $\nu \equiv f(1 + \mathbf{V}_{\rm orb} \cdot \tilde{\mathbf{n}})$.

The varying sky-metric complicates ☞ template placement, ☞ coincidence schemes over the sky and ☞ estimating the number of templates.

"Supersky" metric

We formally relax the constraint $|\tilde{\mathbf{n}}| = 1$, allowing the sky-vector $\mathbf{n}$ to have 3 independent components $\{n^x, n^y, n^z\}$. This 3D-embedding of the 2D sky-sphere greatly simplifies the expressions for the supersky-metric components, namely

$$g_{ll'} = \langle r_l r_{l'} \rangle - \langle r_l \rangle \langle r_{l'} \rangle$$

which is constant over the full extended parameter space for any observation time T. The same is true for the mixed sky-spin components of the metric, which have the form $g_{ls} = \langle r_l t^s \rangle - \langle r_l \rangle \langle t^s \rangle$. The curved sky-metric is obtained by *restricting* the supersky to the 2D sky-sphere:

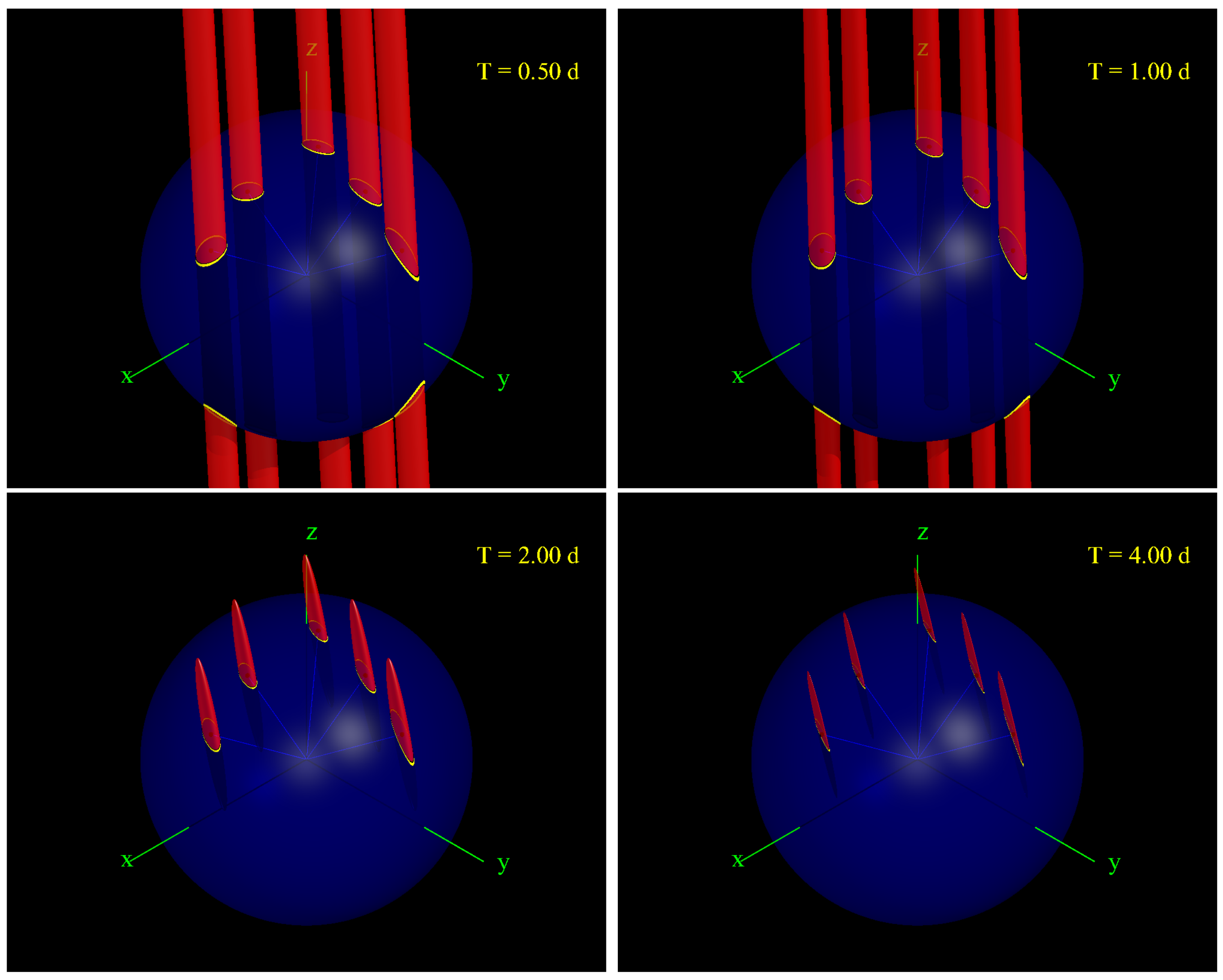


Figure: Supersky iso-mismatch ellipsoids (red) and their restriction (yellow) to the sky-sphere (blue) $|\mathbf{n}| = 1$, for different observation times T, at fixed frequency ν.

Discussion

Supersky-metric is *constant* over extended parameter space!

- ☞ sky-metric obtained from simple projection onto 2D sky-sphere
- ☞ simple coincidence scheme over whole parameter-space
- ☞ allows estimation of number of templates using "flux" formalism

References

[1] Jaranowski, Królak, Schutz, *PRD* **58**, 063001 (1998)
[2] B. Owen, *PRD* **53**, 6749 (1996)
[3] R. Balasubramanian, et al., *PRD* **53**, 3033 (1996)
[4] R. Prix, *PRD* **75**, 023004 (2007)