Towards Gravitational-Wave Asteroseismology

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OUTLINE

- ⋆ Gravitational Waves from Compact Stars
  - ▶ Quasi-normal (damped) oscillation modes
  - ▶ Dependence of the eigenfrequencies on the star structure and dynamics
- ⋆ Gravitational-wave asteroseismology
  - ▶ Determining star properties from the observation of gravitational waves
  - ▶ Exploiting the observation of gravitational waves to costrain theoretical models of the structure and dynamics of compact stars
- ⋆ Summary & Outlook

GRAVITATIONAL WAVES FROM COMPACT STARS

★ Bottom line: compact objects emit GWs at the frequencies of their quasi-normal (QN) oscillation modes

★ The *complex* frequencies of QN modes carry information on the internal structure of the emitting source

  ▷ For black holes they only depend on the parameters specifying space-time geometry: mass, charge and angular momentum
  ▷ For stars, things are far less simple, as the frequencies of QN modes depend on the properties of matter in the star interior, of which little is known

★ Detection of the variety of pulsation modes will provide new information on the internal structure and dynamics of the emitting star, setting stringent constraints on the available theoretical models

PRELIMINARIES

★ Consider a star characterized by a static and spherically symmetric distribution of matter in chemical, hydrostatic and thermodynamic equilibrium

★ The metric of the gravitational field generated by the star can be written in the form ($x^0 = t, x^1 = \varphi, x^2 = r, x^3 = \theta$)

$$ds^2 = g_{\mu\nu}^0 dx^\mu dx^\nu = e^\nu dt^2 - e^\lambda dr^2 - r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right)$$

★ ν and λ are functions of r, to be determined solving Einstein's equations (in geometric units)

$$\begin{aligned} G_{\mu\nu} &= 2\,T_{\mu\nu} \quad &\text{inside the star} \\ G_{\mu\nu} &= 0 \quad &\text{outside the star} \end{aligned}$$

▷ $G_{\mu\nu}$: Einstein's tensor, $T_{\mu\nu}$: energy-momentum tensor

NON-RADIAL OSCILLATIONS

- ⋆ Consider a perturbation inducing a small amplitude motion described by the displacement 3-vector $\xi^i(x)$
- ⋆ Due to the fluid motion the geometry of spacetime is no longer described by the metric tensor $g^0_{\mu\nu}$
- ⋆ In standard notation

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \left(g^0_{\mu\nu} + h_{\mu\nu}\right)dx^\mu dx^\nu = ds_0^2 + h_{\mu\nu}dx^\mu dx^\nu$$

- ⋆ $g^0_{\mu\nu}$ corresponds to the unperturbed state and can be determined from TOV equation
- ⋆ At first order, Einstein's equations become a set of *linear* differential equations linking the thirteen functions $\xi^i(x)$ and $h_{\mu\nu}(x)$
- ⋆ Studying non radial oscillations amounts to determining the solutions of these equations

Solutions of Perturbed Equations

★ Assume that the time dependence of the solutions be of the standard form $\exp i\sigma t$ (linearity + stationary background metric)

★ The radial and angular dependence of the solutions can be separated

▷ Scalar functions can be expanded in ordinary spherical harmonics

$$f(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m}(r) Y_{\ell m}(\theta, \phi)$$

▷ Under parity transformations spherical harmonics transform according to

$$Y_{\ell m}(\pi - \theta, \pi + \phi) = (-1)^{\ell} Y_{\ell m}(\theta, \phi)$$

⋆ Similarly, any quantity transforming as a tensor under rotations , e.g. h^{ij}, can be expanded in *tensorial* spherical harmonics

$$h^{ij} = \sum_{\ell m} \left[\sum_{k=1}^{3} h_{\ell m}^{k,ax}(r) \left\{ A_{\ell m}^{ij} \right\}^k (\theta, \phi) + \sum_{h=1}^{7} h_{\ell m}^{h,pol}(r) \left\{ P_{\ell m}^{ij} \right\}^h (\theta, \phi) \right]$$

⋆ The three tensorial harmonics called *axial* are *odd* and the seven called *polar* are *even*, because under parity transformations they tranform according to

$$\left\{ P_{\ell m}^{ij} \right\}^h (\pi - \theta, \pi + \phi) = (-1)^{\ell} \left\{ P_{\ell m}^{ij} \right\}^h (\theta, \phi)$$

$$\left\{ A_{\ell m}^{ij} \right\}^k (\pi - \theta, \pi + \phi) = (-1)^{\ell+1} \left\{ A_{\ell m}^{ij} \right\}^k (\theta, \phi)$$

⋆ Quasi-normal modes are labelled *polar* or *axial* according to the parity of the corresponding perturbation

A MORE PHYSICAL CLASSIFICATION

* Consider the equation describing the displacement associated with the perturbation in Newtonian theory

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = -\rho_0 \boldsymbol{\nabla} \delta\Phi - \delta\rho \boldsymbol{\nabla} \Phi_0 - \boldsymbol{\nabla} \delta P$$

$$\rho = \rho_0 + \delta\rho \, , \; P = P_0 + \delta P \, , \; \Phi = \Phi_0 + \delta\Phi$$

* In the rhs of the above equation, the *restoring force* consists of three contributions

  ▷ $\rho_0 \boldsymbol{\nabla} \delta\Phi$: change of the gravitational field
  ▷ $\delta\rho \boldsymbol{\nabla} \Phi_0$: change of density (buoyancy)
  ▷ $\boldsymbol{\nabla} \delta P$: gradient of pressure

* Quasi-normal modes are classified according to the prevailing restoring force

GRAVITATIONAL WAVES FROM COMPACT STARS

- ⋆ Recall: a star emits GW at the (complex) frequencies of its QN modes

  - ▶ g-modes: main restoring force is buoyancy
  - ▶ p-modes: main restoring force is pressure
  - ▶ f-mode: intermediate between g- and p-modes
  - ▶ w-modes: pure space-time modes
  - ▶ r-modes: main restoring force is the Coriolis force

$$\{\omega_{gn}\} < \omega_f < \{\omega_{pn}\} < \{\omega_{wn}\}$$

- ⋆ In newtonian theory the frequency of the f-mode is proportional to the average density of the star
- ⋆ g-modes appear in presence of thermal or composition gradients

GW EMISSION & EQUATION OF STATE (EOS)

- ⋆ QN oscillation modes associated with GW emission depend upon the EOS—i.e. the relation linking pressure and energy-density—describing the properties of matter inside the star

- ⋆ Consider the axial (i.e. odd parity) w-modes as an example

  - ▶ Their frequencies are complex eigenvalues of a Scrödinger-like equation, whose "potential" $V_\ell(r)$ explicitly depends on the EoS

$$V_\ell(r) = \frac{e^{2\nu(r)}}{r^3} \left\{ \ell(\ell+1)r + r^3 \left[\epsilon(r) - P(r) \right] - 6M(r) \right\}$$

wiht (recall)

$$\frac{d\nu}{dr} = -\frac{1}{\left[\epsilon(r) + P(r) \right]} \frac{dP}{dr}$$

★ Evolution of $V_\ell(r)$ as a function of inverse compactness ($\sim R/M$)
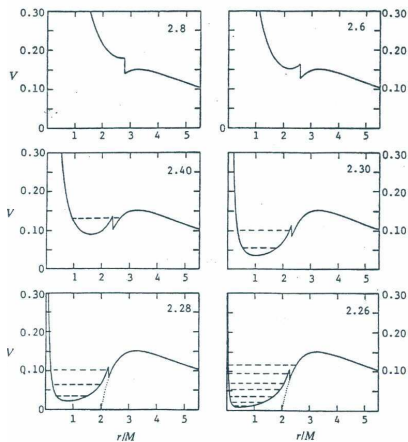
GW ASTEROSEISMOLOGY

- ★ AD 1998: Andersson and Kokkotas herald the advent of GW astereoseismology, declaring that "The day of the first undeniable detection of gravitational waves should not be far away"

Towards gravitational wave asteroseismology

Nils Andersson[1,2] and Kostas D. Kokkotas[3,4]

[1]*Department of Physics, Washington University, St Louis, MO 63130, USA*
[2]*Institut für Theoretische Astrophysik, Universität Tübingen, D-72076 Tübingen, Germany*
[3]*Department of Physics, Aristotle University of Thessaloniki, Thessaloniki 54006, Greece*
[4]*Max-Planck-Institut für Gravitationsphysik, Schlaatzweg 1, 14473 Potsdam, Germany*

- ★ AD 2004: Benhar, Ferrari & Gualtieri argue that astereoseismology is promising to the extent to which compact stars can be modelled at a sufficient level of realism

Gravitational wave asteroseismology reexamined

Omar Benhar,[2,1] Valeria Ferrari,[1,2] and Leonardo Gualtieri[1,2]
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- ⋆ Bottom line: want to exploit the detection of gravitational waves to extract information on the properties of the emitting star (e.g. its radius, or the composition of matter in its interior)
- ⋆ Two possible strategies
    1. Find a set of empirical relations allowing to express the mode frequencies in terms of appropriate scaling variables, largely independent of the choice of EOS → use the detected signal to obtain the star radius knowing its mass
    2. Study the dependence of the pattern of emitted gravitational waves predicted by different stellar models, corresponding to different dynamiccal models and composition) → use the detected signal to constrain the models

EARLY ATTEMPTS WITHIN STRATEGY 1

- Andersson & Kokkotas, 1998
  - ▷ compute the real and imaginary part of the frequencies of the (polar and axial) f-mode and the first p- and w- modes for a variety of EOS
  - ▷ identify the "scaling" variables

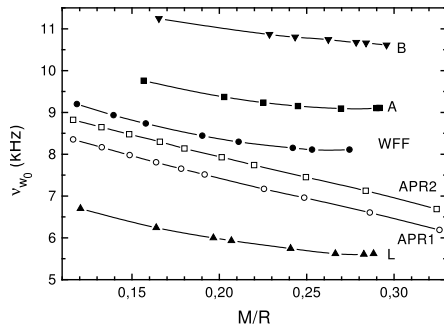$$\frac{\bar{M}}{\bar{R}^3} \ , \ \frac{\bar{M}}{\bar{R}}$$

$$\bar{M} = \frac{M}{1.4 \ M_\theta} \ , \ R = \frac{R}{10 \ \text{km}}$$

- The mode frequencies and damping times, when plotted as a function of the above variables, show little dependence on the choice of EOS

ENERGY AND DAMPING TIME OF THE f-MODE
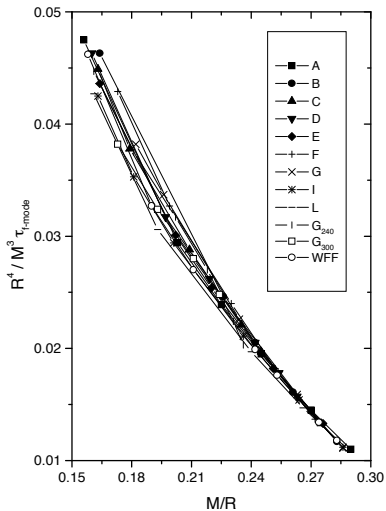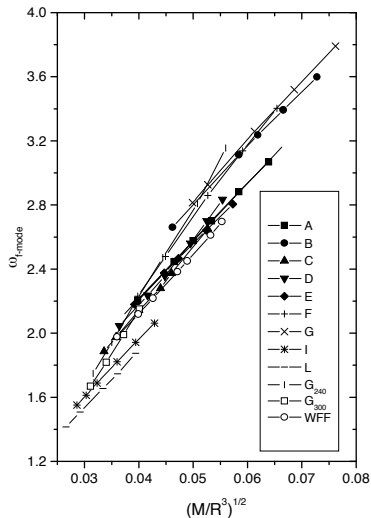
EARLY ATTEMPTS WITHIN STRATEGY 2

▷ frequency of the 1st axial w-mode vs star compactness (OB, Berti & Ferrari, 1999)

▷ the pattern of frequencies strictly reflects the (local) stiffness of the EOS ($\Gamma = d\log P/d\log\rho$). Softer (i.e. lower Γ) EOS correspond to higher frequencies

▷ for a given EOS, the frequency depends weakly upon M/R

Universality Revisited

▷ Frequency of the f-mode computed using different models based on state-of-the-art EOS (OB, Ferrari & Gualtieri, 2004)
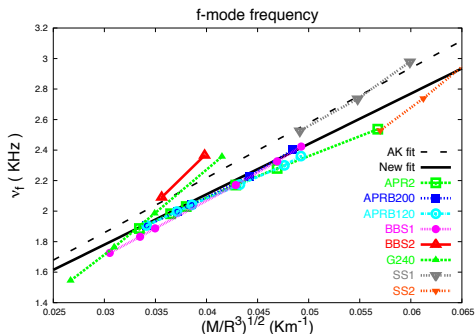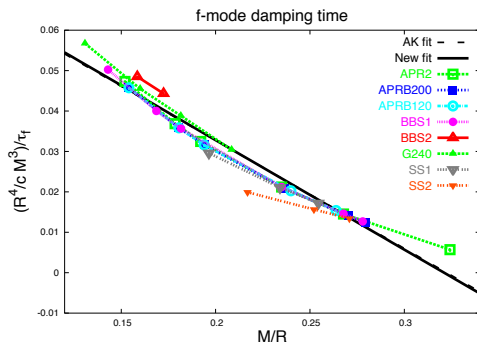

$$\nu_f = a + b\sqrt{\frac{M}{R^3}}$$

$$a = 0.79 \pm 0.09 \text{ kHz}$$

$$b = 33 \pm 2 \text{ km kHz}$$

UNIVERSALITY REVISITED

▷ Damping time of the f-mode computed using different models
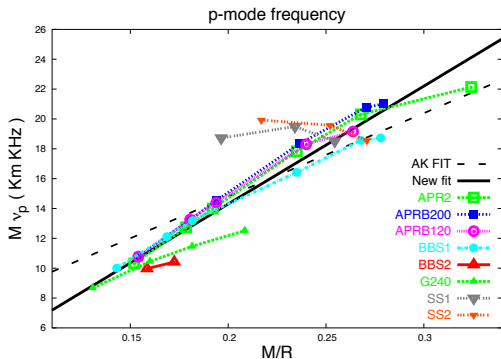based on state-of-the-art EOS (OB, Ferrari & Gualtieri, 2004)

f-mode damping time

$$\tau_f = \frac{R^4}{cM^3} \left[a + b\,\frac{M}{R} \right]^{-1}$$

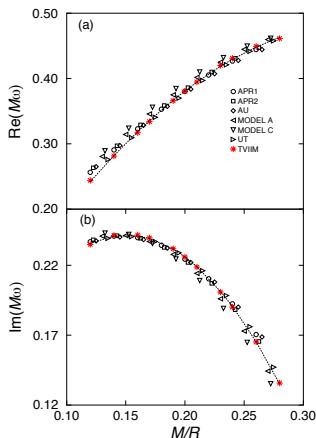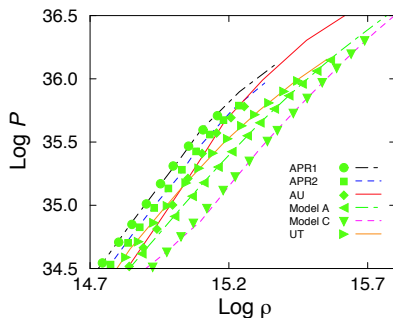$$a = (8.7 \pm 0.2) \times 10^{-2}$$

$$b = -0.271 \pm 0.009$$

UNIVERSALITY REVISITED

▷ Frequency of the first p-mode computed using different models based on state-of-the-art EOS (OB, Ferrari & Gualtieri, 2004)
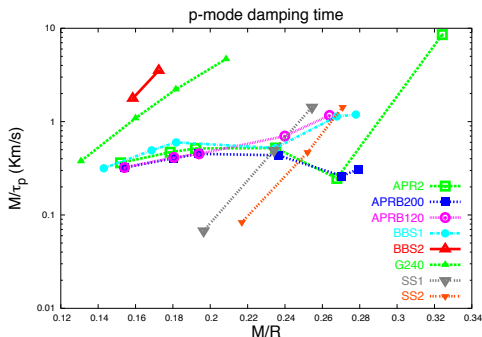

p-mode frequency

UNIVERSALITY REVISITED

▷ Frequencies and damping times of the first w-mode computed
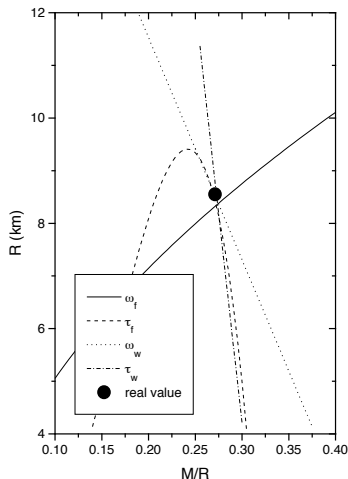using different models of EOS (Tsui & Leung, 2005)

FAILURE OF UNIVERSALITY

▷ Damping time of the first p-mode computed using different models based on state-of-the-art EOS (OB, Ferrari & Gualtieri, 2004)

▷ *universality* does not appear to work here !

p-mode damping time

A NUMERICAL EXPERIMENT (ANDERSSON & KOKKOTAS)

- select a model polytropic star ($P \propto \epsilon^{\Gamma}$ EOS, easily solvable) and compute M and R
- compute frequency and damping time of the f-mode and the 1st w-mode
- plot the four lines corresponding to the empirical relations
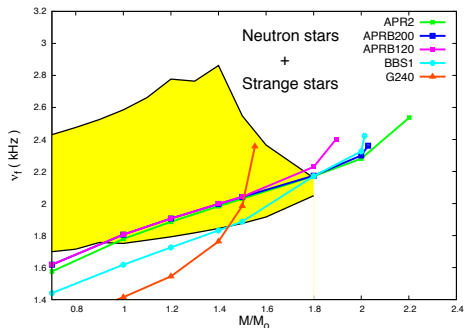- the intersection of the four lines gives the correct M and R with a few percent accuracy

MORE ON STRATEGY 2

▷ f-mode frequency as a function of the neutron star mass
(OB, Ferrari & Gualtieri, 2004)
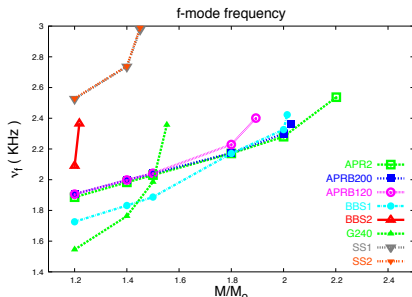
▷ stars containing hyperons and strange stars have much higher frequencies

NEUTRON STARS *vs* STRANGE STARS

★ strange stars have $\nu_f \geq 1.7$ kHz and $M < 1.8\ M_\odot$

★ a wide frequency range corresponds to strange stars only (for example, $\nu_f \geq 1.9$ kHz for a star of mass $M = 1.2\ M_\odot$)

★ frequencies $\nu_f \gtrsim 2.2$ kHz most likely correspond to strange stars, *regardless of the mass*

SUMMARY & OUTLOOK

- ⋆ Oscillations are a promising source of GW emission from compact stars

- ⋆ The rich oscillation spectra could allow to probe the internal structure and dynamics of the star

- ⋆ Realistic modeling of the astrophysical processes leading to the excitation of oscillation modes are needed. Hovever, besides the equation of state, achieving this goal will require the understanding of a variety of properties of neutron star matter, such as the transport coefficients, the occurrence of superfluid and superconducting gaps and the neutrino emission and scattering rates.

EQUILIBRIUM EQUATIONS

★ Assuming that matter inside the star can be described as a perfect fluid characterized by the four-velocity field u^μ

$$T_{\mu\nu} = (\epsilon + P)\, u_\mu u_\nu - P g_{\mu\nu}$$

  ▷ ϵ : energy density , P : pressure

★ From the tt component of Einstein's equations it follows that

$$e^{-\lambda(r)} = 1 - \frac{2}{r}\, M(r)$$

$$M(r) = 4\pi \int_0^r \epsilon(r')r'^2 dr'$$

★ The rr component yields

$$\frac{d\nu(r)}{dr} = -2\left(4\pi P(r)r + \frac{M(r)}{r^2}\right)\left(1 - 2\,\frac{M(r)}{r}\right)^{-1}$$

* From energy-momentum conservation

$$\frac{d\nu(r)}{dr} = -\frac{2}{\epsilon(r) + P(r)}\frac{dP}{dr}$$

* Combining together the above relations leads to the equation describing hydrostatic equilibrium of a spherically symmetric star in general relativity: the Tolman-Oppenheimer-Volkoff equation

$$\frac{dP}{dr} = -\frac{[\epsilon(r) + P(r)]\left[M(r) + 4\pi r^3 P(r)\right]}{r^2\left[1 - 2M(r)/r\right]} \xrightarrow[c \to \infty]{} \frac{\epsilon(r)M(r)}{r^2}$$

* In vacuum $\epsilon = P = 0$ and we find the Schwarzschild solution

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\vartheta d\varphi^2\right)$$

DETECTION OF GW FROM COMPACT STARS

▷ Assume that the f-mode of a neutron star with $\nu_f = 1.9$ kHz, $\tau_f = 0.184$ s has been excited

▷ The signal emitted can be modeled as (Ferrari et al, 2003)

$$h(t) = h_0 \, e^{-(t-t_0)/\tau_f} \sin\left[2\pi\nu_f \left(t - t_0\right)\right] \, ,$$

and the energy stored into the mode is

$$dE_{\text{mode}} = \frac{\pi}{2} \nu^2 \, |\, \tilde{h}(\nu)\,|^2 \, dSd\nu$$

▷ Will the VIRGO interferometer be able to detect this signal ?

DETECTION OF GW FROM COMPACT STARS

▷ VIRGO noise power spectral density

$$S_n(x) = 10^{-46} \cdot \left\{ 3.24[(6.23x)^{-5} + 2x^{-1} + 1 + x^2] \right\} \ \text{Hz}^{-1},$$

with $x = \nu/\nu_0$ and $\nu_0 = 500 \ \text{Hz}$

▷ Signal to noise ratio

$$SNR = 2 \left[\int_0^\infty d\nu \ \frac{|\tilde{h}(\nu)|^2}{S_n(\nu)} \right]^{1/2}$$

▷ $SNR = 5$ requires $E_{\text{mode}} \sim 6 \times 10^{-7} \ M_\odot$ for a source in our galaxy (d ~ 10 kpc) and $\sim 1.3 \ M_\odot$ for a source in the VIRGO cluster (d ~ 15 Mpc)